Black Holes in Non-local Gravity

非局部引力中的黑洞

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In this chapter, we present a status report of black hole-like solutions in non-local theories of gravity in which the Lagrangians are at least quadratic in curvature and contain specific non-polynomial (i.e., non-local) operators. In the absence of exact black hole solutions valid in the whole spacetime, most of the literature on this topic focus on approximate and simplified equations of motion, which could provide insights on the full non-linear solutions. Therefore, the largest part of this chapter is devoted to the linear approximation. We present results on stationary metric solutions (including both static and rotating cases) and dynamical spacetimes describing the formation of nonrotating mini black holes by the collapse of null shells. Non-local effects can regularize the curvature singularities in both scenarios, and, in the dynamical case, there exists a mass gap below which the formation of an apparent horizon can be avoided. In the final part, we discuss interesting attempts toward finding non-linear black hole solutions in non-local gravity. Throughout this chapter, instead of focusing on a particular non-local model, we present results valid for large classes of theories (to a feasible extent). This more general approach allows the comparison of similarities and differences of the various types of non-local gravity models.

本章我们介绍非局部引力理论中类黑洞解的研究现状，这类理论的拉格朗日量至少是曲率二阶，且包含特定的非多项式 (即非局部) 算符。由于目前不存在适用于全时空的精确黑洞解，该领域的多数研究都聚焦于近似简化的运动方程，以期为完整非线性解提供思路。因此，本章大部分内容围绕线性近似展开。我们介绍定态度量解 (包含静态与旋转两种情形) 和描述零壳坍缩形成无旋转小黑洞的动态时空的研究结果。在这两类情形中，非局部效应都可以正则化曲率奇点；而在动态情形中，存在一个质量间隙，低于该间隙时可以避免形成表观视界。在本章最后部分，我们讨论在非局部引力中寻找非线性黑洞解的若干重要尝试。在全章中，我们没有仅聚焦于某一特定非局部模型，而是在可行范围内呈现适用于大范围理论类的结果。这种更具一般性的研究方式可以方便我们对比各类非局部引力模型的异同。

Keywords

关键词

Black holes · Non-local gravity · Ghost-free gravity · Curvature singularity · Regular solutions · Non-locality · Infinite derivatives

黑洞 · 非局域引力 · 无鬼引力 · 曲率奇点 · 正则解 · 非局域性 · 无穷阶导数

Introduction

引言

Einstein's general relativity (GR) has been the most successful theory to describe classical aspects of gravity so far as many of its predictions have been confirmed to high precision [76]. Despite its great achievements, there are still unsolved problems suggesting that GR can only be used as an effective field theory of gravitational interaction, which works very well at low energy but breaks down in the ultraviolet (UV) regime. In fact, at the classical level, GR is plagued by the presence of black hole and cosmological singularities [46], while at the quantum level, the theory is perturbatively non-renormalizable [1, 45] .

爱因斯坦广义相对论 (GR) 是迄今为止描述引力经典性质最成功的理论，它的诸多预言都得到了高精度验证 [76]。尽管广义相对论成果斐然，仍有不少未解问题表明，它只能作为引力相互作用的有效场论: 该理论在低能区表现极佳，在紫外 (UV) 区却会失效。事实上，在经典层面，广义相对论受黑洞与宇宙学奇点问题困扰 [46]；在量子层面，该理论在微扰下不可重整 [1, 45] 。

A natural way to address these issues is to extend GR by adding higher-curvature terms to the Einstein-Hilbert action. Remarkably, a theory of gravity quadratic in curvature can be perturbatively renormalizable in four dimensions [71]. However, this higher-derivative theory turns out to be pathological, because of a massive spin- 2 ghost-like degree of freedom that causes instabilities and breaks the S -matrix unitarity.

解决这些问题的一种自然思路，是在爱因斯坦-希尔伯特作用量中加入高曲率项，以此推广广义相对论。值得注意的是，曲率二次型引力理论在四维空间中可以实现微扰可重整 [71]。但这种高阶导数理论存在缺陷: 它存在一个有质量自旋 2 类鬼自由度，会引发不稳定性，破坏 S 矩阵幺正性。

This type of unhealthy degree of freedom can be avoided if the action is non-polynomial in field derivatives. Indeed, by considering quadratic-curvature terms that contain specific infinite-derivative operators, one can prevent the appearance of extra pathological modes in the physical spectrum, and still have an improved propagator in the UV, making it possible to formulate theories of gravity that are ghost-free and renormalizable [11, 13, 58, 59, 61, 73]. The presence of non-polynomial differential operators renders the gravitational action non-local.

如果作用量在场导数层面是非多项式的，就可以避免这类不健康的自由度。实际上，通过引入包含特定无限阶导数算符的二次曲率项，可以阻止物理谱中出现额外病态模式，同时仍能在紫外区获得改进的传播子，从而有可能构造出无鬼且可重整的引力理论 [11, 13, 58, 59, 61, 73]。非多项式微分算符的存在使得引力作用量具有非局域性。

In this chapter, we discuss black hole solutions and alternatives in the context of non-local theories of gravity that are UV modifications of GR. The presentation has the form of a status report on this topic and it reviews the most relevant results obtained so far, providing the essential details of both conceptual and computational aspects. We emphasize that, owing to the difficulties involved, a rigorous study of exact black hole solutions in these models is still lacking. Therefore, the results compiled here mainly consist of approximate solutions valid in certain regimes, which might be useful to understand the behavior of the full non-linear ones.

本章我们在作为广义相对论紫外修正的非局域引力理论框架下，讨论黑洞解及相关研究。本章以该领域研究进展综述的形式呈现，梳理了迄今为止得到的最重要的成果，同时涵盖概念与计算两方面的核心细节。我们需要说明，受相关问题本身的难度限制，目前仍缺乏对这些模型中精确黑洞解的严格研究。因此，本文整理的结果主要是适用于特定区域的近似解，这些结果或许有助于理解完整非线性解的行为。

The chapter is organized as follows:

本章结构安排如下:

Section ”Non-local Theories of Gravity”: Since most of the considerations are carried out in the linear regime, in this section, we briefly review the linearized formulation of ghost-free non-local theories of gravity and show several interesting models that have been intensively studied in the literature.

“非局域引力理论”节: 由于大部分讨论在线性区展开，本节我们简要回顾无鬼非局域引力的线性化表述，介绍文献中已被深入研究的若干有趣模型。

Section ”Stationary Linearized Solutions”: We discuss linearized stationary solutions in two situations, namely, static and rotating cases. We show that non-locality can regularize both point-like and ringlike singularities and make several remarks about the physical implications. In what concerns the resolution of point-like singularities, we show that this regularization can occur at different levels (i.e., potential, curvature invariants, and curvature-derivative invariants), depending on the type of form factor used.

“稳态线性化解”节: 我们分静态和转动两种情况讨论线性化稳态解。我们指出非局域性可以正则化点状奇点和环状奇点，并对其物理意义做了若干讨论。关于点状奇点的解决，我们说明，正则化可以在不同层面实现 (即势、曲率不变量和曲率导数不变量)，具体取决于所用形状因子的类型。

Section ”Mini Black Hole Formation by the Collapse of Null Shells”: The dynamical solution of mini black hole formation by the collapse of null shells is discussed. We consider the cases of thin and thick shells. In both cases, there exists a mass gap below which the formation of an apparent horizon can be prevented, while the curvature singularities can also be regularized if the imploding shell has some nonvanishing thickness.

“零壳坍缩形成迷你黑洞”节: 我们讨论零壳坍缩形成迷你黑洞的动力学解，分别考虑薄壳和厚壳两种情况。两种情况下都存在质量间隙: 质量低于该间隙时不会形成表观视界；若坍缩壳存在非零厚度，曲率奇点也可以被正则化。

Section ”Toward the Non-linear Regime”: We discuss the difficulties introduced by non-linearities in the field equations and review an interesting attempt toward understanding the effects of non-locality at the non-linear level. By working in a simplified situation, it is possible to obtain regular black hole solutions with a mass gap and with multiple horizons.

“走向非线性区”节: 我们讨论场方程中非线性项带来的困难，梳理了一个在非线性层面理解非局域效应的重要研究工作。在简化假设下，可以得到带有质量间隙和多 horizons 的正则黑洞解。

Section ”Concluding Remarks”: We present our concluding remarks.

“结语”节: 我们给出本章的总结性讨论。

Throughout this chapter, we use the mostly plus metric convention, with the Minkowski metric $\eta_{\mu\nu} = \mathrm{diag}(-1,+1,+1,+1)$. The Riemann curvature tensor is defined by

贯穿全章，我们采用号差主要为正的度规约定，闵氏度规为 $\eta_{\mu\nu} = \mathrm{diag}(-1,+1,+1,+1)$ 。黎曼曲率张量定义为

$$R^{\alpha}{}_{\beta\mu\nu} = \partial_\mu \Gamma^{\alpha}_{\beta\nu} - \partial_\nu \Gamma^{\alpha}_{\beta\mu} + \Gamma^{\alpha}_{\mu\tau}\Gamma^{\tau}_{\beta\nu} - \Gamma^{\alpha}_{\nu\tau}\Gamma^{\tau}_{\beta\mu}, \tag{1}$$

while $R_{\mu\nu} = R^{\alpha}{}_{\mu\alpha\nu}$ and $R = g^{\mu\nu}R_{\mu\nu}$ are, respectively, the Ricci tensor and the scalar curvature. Also, we adopt the unit system such that $c = 1$ and $\hbar = 1$.

其中 $R_{\mu\nu} = R^{\alpha}{}_{\mu\alpha\nu}$ 和 $R = g^{\mu\nu}R_{\mu\nu}$ 分别为里奇张量和标量曲率。此外，我们采用单位制，满足 $c = 1$ 和 $\hbar = 1$ 。

Non-local Theories of Gravity

引力非局部理论

Einstein’s GR is described in terms of a very simple Lagrangian which is linear in curvature. A natural way of modifying the theory in the UV regime is to generalize the Einstein-Hilbert action through the inclusion of higher-order curvature invariants which contain higher derivatives. Local higher-derivative theories of gravity are usually considered pathological because of the ghost-like degrees of freedom that occur in the spectrum, in the conventional quantum field theory framework. However, as we discuss in this section, ghosts can be avoided if the principle of locality is given up.

爱因斯坦广义相对论由一个极其简单的拉格朗日量描述，该拉格朗日量对曲率是线性的。在紫外区修改该理论的一种自然方式，是通过引入含高阶导数的高阶曲率不变量来推广爱因斯坦-希尔伯特作用量。在常规量子场论框架下，局部高阶导数引力理论通常被认为存在问题，因为其谱中会出现类鬼自由度。但正如我们在本节讨论的，若放弃局域性原理就可以避免鬼的出现。

Let us consider the generic action quadratic in curvature:

我们来考虑曲率二次型的一般作用量:

$$S=\frac{1}{2\kappa^2}\int \mathrm{d}^4x\sqrt{-g}\Big\{R+\frac{1}{2}\big[R\mathcal{F}_1(\Box)R+R_{\mu\upsilon}\mathcal{F}_2(\Box)R^{\mu\upsilon}$$

$$+R_{\mu\upsilon\rho\sigma}\mathcal{F}_3(\Box)R^{\mu\upsilon\rho\sigma}\big]\Big\}, \tag{2}$$

where $\kappa=\sqrt{8\pi G}$, G is Newton’s constant and the form factors $\mathcal{F}_i(\Box)$ are nonlocal, i.e., non-polynomial functions of the d’Alembertian $\Box=g^{\mu\upsilon}\nabla_\mu\nabla_\upsilon$. In particular, we assume that $\mathcal{F}_i(z)$ are analytic functions around $z=0$, in order to guarantee a smooth and consistent infrared limit. Therefore, expanding them in Taylor series,

其中 $\kappa=\sqrt{8\pi G}$, G 是牛顿常数，形状因子 $\mathcal{F}_i(\Box)$ 是非局部的，即达朗贝尔算符 $\Box=g^{\mu\upsilon}\nabla_\mu\nabla_\upsilon$ 的非多项式函数。为保证红外极限光滑自洽，我们特别假设 $\mathcal{F}_i(z)$ 在 $z=0$ 附近是解析函数。因此，将它们展开为泰勒级数,

$$\mathcal{F}_i(\Box)=\sum_{n=0}^{\infty}f_{i,n}\Box^n,\ i=1,2,3, \tag{3}$$

and using the identity (see, e.g., [6, 33])

再利用恒等式 (例如参见文献 [6, 33])

$$R_{\mu\nu\rho\sigma}\Box^nR^{\mu\nu\rho\sigma}=4R_{\mu\nu}\Box^nR^{\mu\nu}-R\Box^nR+O\left(R^3_{\dots}\right)+\mathrm{div}, \tag{4}$$

the action (2) can be rewritten as

作用量 (2) 可以改写为

$$S=\frac{1}{2\kappa^2}\int \mathrm{d}^4x\sqrt{-g}\Big\{R+\frac{1}{2}\big[RF_1(\Box)R+R_{\mu\nu}F_2(\Box)R^{\mu\nu}+O\big(R^3_{\dots}\big)\big]\Big\}, \tag{5}$$

where

其中

$$F_1(\Box)=\mathcal{F}_1(\Box)-\mathcal{F}_3(\Box),\ F_2(\Box)=\mathcal{F}_2(\Box)+4\mathcal{F}_3(\Box), \tag{6}$$

and we have omitted boundary terms.

我们已经省略了边界项。

Since most part of this chapter is based on the linearized equations of motion, which come from the second-order metric perturbations around the Minkowski spacetime at action level, we neglect the cubic-curvature terms $O\left(R^3\right)$ in (5) (or, in other words, the Riemann-squared term $R_{\mu\upsilon\rho\sigma}\mathcal{F}_3\left(\Box\right)R^{\mu\upsilon\rho\sigma}$). Therefore, in what follows, we analyze the gravitational action:

由于本章大部分内容基于线性化运动方程——该方程由作用量层面闵氏时空附近的二阶度量扰动得到——我们忽略 (5) 中的三次曲率项 $O\left(R^3\right)$ (换句话说，即黎曼平方项 $R_{\mu\upsilon\rho\sigma}\mathcal{F}_3\left(\Box\right)R^{\mu\upsilon\rho\sigma}$)。因此，下文我们将分析如下引力作用量:

$$S=\frac{1}{2\kappa^2}\int \mathrm{d}^4x\sqrt{-g}\left\{R+\frac{1}{2}\left[RF_1\left(\Box\right)R+R_{\mu\upsilon}F_2\left(\Box\right)R^{\mu\upsilon}\right]\right\}. \tag{7}$$

Linearized Theory

线性化理论

In the weak gravitational field approximation, we consider linear perturbations around the Minkowski background,

在弱引力场近似下，我们考虑闵氏背景下的线性微扰，

$$g_{\mu\upsilon}=\eta_{\mu\upsilon}+\kappa h_{\mu\upsilon}, \tag{8}$$

and expand the action (7) up to order $O\left(h^2_{\dots}\right)$ [13, 37, 61] , namely,

并将作用量 (7) 展开至 $O\left(h^2_{\dots}\right)$ [13, 37, 61] 阶，即

$$\begin{aligned}S^{(2)}=\frac{1}{4}\int \mathrm{d}^4x\Big\{&\frac{1}{2}h_{\mu\nu}a\left(\Box\right)\Box h^{\mu\nu}-h^\sigma_\mu a\left(\Box\right)\partial_\sigma\partial_\nu h^{\mu\nu}+hc\left(\Box\right)\partial_\mu\partial_\nu h^{\mu\nu}\\&-\frac{1}{2}hc\left(\Box\right)\Box h+\frac{1}{2}h^{\lambda\sigma}\frac{a\left(\Box\right)-c\left(\Box\right)}{\Box}\partial_\lambda\partial_\sigma\partial_\mu\partial_\nu h^{\mu\upsilon}\Big\},\end{aligned} \tag{9}$$

where

其中

$$a\left(\Box\right)\equiv 1+\frac{1}{2}F_2\left(\Box\right)\Box, \tag{10}$$

$$c\left(\Box\right)\equiv 1-2F_1\left(\Box\right)\Box-\frac{1}{2}F_2\left(\Box\right)\Box, \tag{11}$$

and $h\equiv\eta_{\mu\upsilon}h^{\mu\upsilon}$ is the trace of the field perturbation. From Eq. (9) we can derive the linearized field equations,

且 $h \equiv \eta_{\mu\nu} h^{\mu\nu}$ 是场微扰的迹。由式 (9) 我们可以推导出线性化场方程,

$$a(\Box)\left(\Box h_{\mu\nu} - \partial_\sigma \partial_\nu h^\sigma_\mu - \partial_\sigma \partial_\mu h^\sigma_\nu\right) + c(\Box)\left(\eta_{\mu\nu}\partial_\rho\partial_\sigma h^{\rho\sigma} + \partial_\mu\partial_\nu h - \eta_{\mu\nu}\Box h\right)$$
$$+\frac{a(\Box) - c(\Box)}{\Box}\partial_\mu\partial_\nu\partial_\rho\partial_\sigma h^{\rho\sigma} = -2\kappa T_{\mu\nu}, \tag{12}$$

where the coupling to matter is introduced through the energy-momentum tensor:

其中通过能量动量张量引入了与物质的耦合:

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta S_{\mathrm{m}}}{\delta g^{\mu\nu}} \simeq \frac{2}{\kappa}\frac{\delta S_{\mathrm{m}}}{\delta h^{\mu\nu}}. \tag{13}$$

In Eq. (13) S_{m} is the matter action. The energy-momentum tensor satisfies the conservation law $\partial_\mu T^{\mu\nu} = 0$, consistently with the Bianchi identity.

式 (13) 中 S_{m} 是物质作用量。能量动量张量满足守恒律 $\partial_\mu T^{\mu\nu} = 0$ ，与比安基恒相自洽。

Propagator

传播子

The propagator around the Minkowski background is computed by inverting the kinetic operator in the Lagrangian (9). Its gauge-fixing independent part reads [3, 13, 61, 73]

闵氏背景下的传播子通过对拉格朗日量 (9) 中的动能算子求逆计算得到，其规范无关部分为 [3, 13, 61, 73]

$$\Pi_{\mu\nu\rho\sigma}(k) = \frac{P^{(2)}_{\mu\nu\rho\sigma}}{k^2 f_2(-k^2)} - \frac{P^{(0-s)}_{\mu\nu\rho\sigma}}{2k^2 f_0(-k^2)}, \tag{14}$$

where

其中

$$f_2\left(-k^2\right) \equiv a\left(-k^2\right),\ f_0\left(-k^2\right) \equiv \frac{3c\left(-k^2\right) - a\left(-k^2\right)}{2}, \tag{15}$$

and $P^{(2,0-s)}_{\mu\nu\rho\sigma}$ are the Barnes-Rivers operators that project the metric fluctuation $h_{\mu\nu}$ along its gauge-independent spin- 2 and spin-0 components [10, 69, 75] .

和 $P^{(2,0-s)}_{\mu\nu\rho\sigma}$ 是 Barnes-Rivers 算子，用于将度规涨落 $h_{\mu\nu}$ 投影到其规范无关的自旋 2 分量和自旋 0 分量 [10, 69, 75] 上。

In general, if the functions $f_{0,2}\left(-k^2\right)$ are polynomials (or, equivalently, if the form factors $F_i\left(\Box\right)$ are polynomials), the propagator (14) possesses, besides the massless graviton, extra massive degrees of freedom that can cause instabilities. This happens because some of these modes are associated with real poles with negative residue, i.e., they are unhealthy ghosts.

一般而言，若函数 $f_{0,2}\left(-k^2\right)$ 是多项式，即等价于形状因子 $F_i\left(\Box\right)$ 是多项式，那么传播子 (14) 除无质量引力子外还存在额外的大质量自由度，这会引发不稳定性。这种情况的成因是部分此类模式对应留数为负的实极点，也就是非物理鬼场。

However, in the framework of non-local models, it is possible to construct classes of ghost-free theories by imposing some requirements on the form factors. Indeed, if

但在非局域模型的框架下，我们可以通过对形状因子施加特定条件来构造无鬼理论的分类。实际上，若

$$f_s\left(-k^2\right) = e^{\gamma_s\left(-k^2\right)},\; s = 0, 2, \tag{16}$$

where $\gamma_2\left(-k^2\right)$ and $\gamma_0\left(-k^2\right)$ are the entire functions, the propagator only has the pole at $k^2 = 0$ (corresponding to the graviton) and no ghost-like degree of freedom. It is worth mentioning that, given the sign difference between the spin-2 and spin-0 parts of the propagator (14), it is possible to introduce an additional healthy massive scalar in the theory, which has important applications in cosmology $[12, 13, 53, 55-57]$.

其中 $\gamma_2\left(-k^2\right)$ 和 $\gamma_0\left(-k^2\right)$ 是整函数，传播子仅在 $k^2 = 0$ 处存在极点 (对应引力子)，不存在鬼类自由度。值得一提的是，由于传播子 (14) 的自旋 2 部分与自旋 0 部分存在符号差，我们可以在理论中引入一个额外的健康有质量标量场，这在宇宙学中有重要应用 $[12, 13, 53, 55-57]$ 。

From Eq. (16) we can also obtain the corresponding relations for the form factors:

由式 (16) 我们还可以得到形状因子满足的对应关系:

$$F_2\left(\Box\right) = 2\frac{e^{\gamma_2(\Box)}-1}{\Box},\; F_1\left(\Box\right) = -\frac{1}{3}\left[\frac{e^{\gamma_0(\Box)}-1}{\Box} + F_2\left(\Box\right)\right]. \tag{17}$$

Let us emphasize that the key role is played by the functions γ_2 and γ_0 , which are entire; thus, the exponentials do not introduce any unhealthy pole. Therefore, non-locality can help to resolve the ghost problem in higher-derivative gravity by means of the introduction of non-polynomial (infinite-derivative) differential operators in the action.

需要强调的是，γ_2 和 γ_0 作为整函数起到了关键作用，因此指数项不会引入任何非物理极点。由此可见，非局域性可以通过在作用量中引入非多项式 (无限导数) 微分算子，帮助解决高阶导数引力中的鬼问题。

Finally, some simplifications can be achieved for the particular case in which $\gamma_2 = \gamma_0 \equiv \gamma$. In this situation, the following condition holds true:

最后，当满足 $\gamma_2 = \gamma_0 \equiv \gamma$ 的特殊情况时可以得到一些简化，此时下述关系成立:

$$f_2(\Box) = f_0(\Box) \Rightarrow F_1(\Box) = -\frac{1}{2}F_2(\Box) = \frac{1-e^{\gamma(\Box)}}{\Box}. \tag{18}$$

Then, the non-local gravity action corresponding to the choice (18) is given by

那么，对应选择 (18) 的非局域引力作用量为

$$S = \frac{1}{2\kappa^2}\int \mathrm{d}^4x\sqrt{-g}\left[R + G_{\mu\nu}\frac{e^{\gamma(\Box)}-1}{\Box}R^{\mu\nu}\right], \tag{19}$$

where $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R$ is the Einstein tensor.

其中 $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R$ 是爱因斯坦张量。

Examples of Form Factors

形状因子示例

Here we review some explicit choices for the entire functions $\gamma_s(\Box)$ that have been intensively studied in the literature.

在此我们梳理文献中已被深入研究的几个整函数 $\gamma_s(\Box)$ 的明确选择。

1. Gaussian form factor. This is the simplest possible choice, corresponding to the entire functions $\gamma_s(-k^2)$ that are monomials of degree one [11, 13, 61] :

1. 高斯形状因子。这是最简单的选择，对应一次单项式的整函数 $\gamma_s(-k^2)$ [11, 13, 61] :

$$\gamma_s(-k^2) = \frac{k^2}{\mu_s^2},\ s = 2, 0, \tag{20}$$

where μ_s represents the physical energy scale at which non-local effects are expected to become important.

其中 μ_s 表示非局部效应预计会变得显著的物理能量标度。

2. Generic exponential form factor. One can generalize the previous example to a generic monomial of degree $n \geq 1$ [24, 34, 37, 58, 68] :

2. 通用指数形状因子。我们可以将上一个例子推广为次数为 $n \geq 1$ [24, 34, 37, 58, 68] 的一般单项式:

$$\gamma_s(-k^2) = \left(\frac{k^2}{\mu_s^2}\right)^n,\ s = 2, 0. \tag{21}$$

The particular case $n = 2$ is sometimes referred to as the Krasnikov form factor, as it was applied for the first time in gravity in Ref. [58].

当 $n=2$ 取特定值时，该形状因子常被称为克拉斯尼科夫形状因子，因为它首次被应用于引力研究是在文献 [58] 中。

3. Kuz'min and Tomboulis form factors. This family of form factors was initially studied in Refs. [59,73] and more recently in Ref. [61] and subsequent works. A general form factor of this type can be written as

3. 库兹明和通布利斯形状因子。这一族形状因子最初在文献 [59,73] 中被研究，近年又在文献 [61] 及后续工作中被讨论。这类形状因子的一般形式可以写为

$$\gamma_s\left(-k^2\right)=\lambda_s\left[\Gamma\left(0,P_s\left(-k^2\right)\right)+\gamma_{\mathrm{E}}+\ln\left(P_s\left(-k^2\right)\right)\right],\ s=2,0, \tag{22}$$

where λ_s is a positive constant, γ_{E} denotes the Euler-Mascheroni constant, $\Gamma(0,z)$ is the incomplete gamma function, and $P_s(z)$ is a real polynomial such that $\gamma_s(0)=0$. The Kuz' min form factor [59] consists in the choice $P_s(z)=-z/\mu_s^2$ with an arbitrary $\lambda_s\in\mathbb{N}$, while the Tomboulis form factor [73] is characterized by $\lambda_s=1/2$ and $P_s(z)$ of arbitrary (but even) degree $N_s\geq 2$.

其中 λ_s 为正的常数，γ_{E} 代表欧拉-马歇罗尼常数，$\Gamma(0,z)$ 是不完全伽马函数，$P_s(z)$ 是满足 $\gamma_s(0)=0$ 的实多项式。库兹明形状因子 [59] 取 $P_s(z)=-z/\mu_s^2$ ，$\lambda_s\in\mathbb{N}$ 任意，而通布利斯形状因子 [73] 满足 $\lambda_s=1/2$ ，且 $P_s(z)$ 为任意偶次多项式 $N_s\geq 2$ 。

Despite their more complicated expressions (in comparison to the previous ones), they are useful to formulate super-renormalizable non-local theories of gravity because, for large momentum, they behave like a polynomial. In fact (22) yields

尽管和前文的形状因子相比，它们的表达式更复杂，但这类形状因子可用于构造超可重整化的非局部引力理论，因为它们在大动量下的行为和多项式一致。实际上由式 (22) 可得

$$e^{\gamma_s(-k^2)}\underset{k^2\to\infty}{\sim}k^{2\lambda_s N_s} \tag{23}$$

where N_s is the degree of the polynomial $P_s(z)$.

其中 N_s 是多项式 $P_s(z)$ 的次数。

Stationary Linearized Solutions

定态线性化解

Most of the solutions in non-local gravity were obtained in the linear approximation. In this section, we describe two types of such solutions, namely, the Newtonian-limit solution and the weak-field slowly rotating stationary solution. To this end, let us consider a generic metric with line element

非局部引力中的大多数解都是在线性近似下得到的。在本节中，我们将介绍这类解的两种类型，即牛顿极限解和弱场慢旋转定态解。为此，我们考虑具有以下线元的一般度规

$$\mathrm{d}s^2 = -(1+2\varphi)\,\mathrm{d}t^2 + 2\vec{h}\cdot\mathrm{d}\vec{r}\,\mathrm{d}t + (1-2\psi)\left(\mathrm{d}x^2+\mathrm{d}y^2+\mathrm{d}z^2\right), \tag{24}$$

where $\vec{r} = (x, y, z)$ is the radius vector and $\varphi = -\kappa h_{00}/2, \psi = -\kappa h_{ii}/2$ and $h_i = \kappa h_{0i}$ are the potentials sourced by $T_{\mu\nu}$.

其中 $\vec{r} = (x, y, z)$ 是径向矢量，$\varphi = -\kappa h_{00}/2, \psi = -\kappa h_{ii}/2$ 和 $h_i = \kappa h_{0i}$ 是由 $T_{\mu\nu}$ 产生的势。

In the nonrelativistic limit, the source is pressureless, $T \equiv \eta^{\mu\upsilon} T_{\mu\upsilon} \simeq -T_{00}$, and its only nonvanishing components are T_{00} and, possibly, T_{0i} . The latter is zero for static point-like sources (so that in this case, $h_i \equiv 0$) and it is nonzero for a rotating dust.

在非相对论极限下，源是无压强的 $T \equiv \eta^{\mu\upsilon} T_{\mu\upsilon} \simeq -T_{00}$ ，它仅有的非零分量为 T_{00} ，还可能包含 T_{0i} 。对于静态点源，后者为零 (因此在这种情况下 $h_i \equiv 0$)，而对于旋转尘埃，它非零。

The metric potentials φ and ψ can be obtained from the 00-component and the trace of the linearized field equations (12), which are equivalent to the set of coupled differential equations:

度规势 φ 和 ψ 可以从线性化场方程 (12) 的 00 分量和迹得到，二者等价于下述耦合微分方程组:

$$[a(\Delta) - c(\Delta)]\,\Delta\varphi + 2c(\Delta)\,\Delta\psi = \kappa^2 T_{00}, \tag{25a}$$

$$[3c(\Delta) - a(\Delta)]\,[2\Delta\varphi - \Delta\psi] = \kappa^2 T_{00}, \tag{25b}$$

$$a(\Delta)\,\Delta h_i = -2\kappa^2 T_{0i}, \tag{25c}$$

where the functions a and c now depend on the spatial Laplacian Δ , as $\Box \simeq \Delta$ for stationary fields (In the case of $T_{0i} \neq 0$, the de Donder gauge $\partial_\mu h^{\mu\upsilon} = 0$ is assumed, or a suitable higher-order generalization compatible with the metric (24) (see, e.g., [4, 18, 23, 72]).). Notice that for $a = c = 1$ (i.e., $F_1 = F_2 = 0$) the system (25) reduces to standard Poisson equations, consistently recovering Einstein's GR result.

其中函数 a 和 c 现在依赖于空间拉普拉斯算符 Δ ，这是定态场的 $\Box \simeq \Delta$ (在 $T_{0i} \neq 0$ 的情况下，假定满足德东规范 $\partial_\mu h^{\mu\upsilon} = 0$ ，或是与度规 (24) 相容的合适高阶推广，参见例如 [4, 18, 23, 72])。注意，对于 $a = c = 1$ (即 $F_1 = F_2 = 0$)，方程组 (25) 退化为标准泊松方程，一致地还原出爱因斯坦广义相对论的结果。

The first two equations in (25) can be decoupled by using the auxiliary potentials [40]:

(25) 中的前两个方程可以通过引入辅助势退耦 [40]:

$$\chi_2 \equiv \frac{\varphi+\psi}{2} \quad \text{and} \quad \chi_0 \equiv 2\psi - \varphi. \tag{26}$$

In fact, the field equations (25) are equivalent to

事实上，场方程 (25) 等价于

$$f_s(\Delta)\,\Delta\chi_s = \frac{\kappa^2}{2}T_{00}, \tag{27}$$

$$f_2(\Delta)\,\Delta h_i = -2\kappa^2 T_{0i}, \tag{28}$$

where $s = 0, 2$ and the functions f_s were defined in Eq. (15). Once the above equations are solved for $\chi_{0,2}$, the original potentials are obtained through the relations:

其中 $s = 0, 2$ 和函数 f_s 已在式 (15) 中定义。上述方程解出 $\chi_{0,2}$ 后，就可以通过以下关系得到原势:

$$\varphi = \frac{4}{3}\chi_2 - \frac{1}{3}\chi_0,\ \psi = \frac{2}{3}\chi_2 + \frac{1}{3}\chi_0. \tag{29}$$

The functions χ_s are called spin- s potentials because they only depend on the spin-s part of the field. In fact, each χ_s only depends on $f_s(\Delta)$, which in turn is related to the gauge-invariant spin- s part of the propagator (14). Also, notice that for the particular case described in (18), we have $a = c = f_2 = f_0$ and all the Newtonian potentials are equal, namely,

函数 χ_s 被称为自旋- s 势，因为它们仅依赖于场的自旋 s 部分。实际上，每个 χ_s 仅依赖于 $f_s(\Delta)$ ，而 $f_s(\Delta)$ 又和传播子 (14) 的规范不变自旋- s 部分相关。此外注意，对于 (18) 中描述的特殊情况，我们有 $a = c = f_2 = f_0$ ，且所有牛顿势都相等，即:

$$\varphi = \psi = \chi_2 = \chi_0 \tag{30}$$

In what concerns the off-diagonal components h_i of the metric, as Eq. (28) only depends on f_2 , they are not affected by F_1 in the linear regime. This is due to the fact that the form factor F_1 modifies only the scalar part of the propagator, which couples to the trace of the energy-momentum tensor. Thus, since the components T_{0i} do not enter in the trace $g^{\mu\upsilon}T_{\mu\upsilon}$, they cannot act as sources for h_i at this order in perturbation.

关于度规的非对角分量 h_i ，由于式 (28) 仅依赖于 f_2 ，它们在线性区不受 F_1 影响。这是因为形状因子 F_1 仅修改传播子的标量部分，而标量部分与能量动量张量的迹耦合。因此，由于分量 T_{0i} 不进入迹 $g^{\mu\upsilon}T_{\mu\upsilon}$ ，它们无法在微扰的这个阶上成为 h_i 的源。

In what follows, we show some general results on the solutions for the Newtonian potentials χ_s , and we postpone the consideration of the off-diagonal components h_i to section ”Resolution of Ringlike Singularities.”

在下文中，我们给出牛顿势 χ_s 解的一些一般结果，而将非对角分量 h_i 的讨论推迟到“类环奇点的消解”一节。

Solution for the Potentials in Terms of Effective Delta Sources

利用有效 δ 源求解引力势

It is possible to solve Eq. (27) directly using the Fourier transform method, or the Laplace transform (see section ”Heat Kernel Solution for the Newtonian Potentials” below), but in some cases, it is instructive to consider an intermediate step, which we describe here (see also, e.g., [27,41]).

我们可以直接利用傅里叶变换法或拉普拉斯变换求解式 (27)(参见下文“牛顿势的热核解”一节)，但在部分场景中，分析一个中间步骤会更有助于理解，我们将在此展开说明 (另可参见文献 [27,41])。

An alternative interpretation of Eq. (27) follows from the inversion of the operator $f_s(\Delta)$, so that one obtains a standard Poisson equation with a modified source:

对式 (27) 的另一种解读源自对算符 $f_s(\Delta)$ 求逆，由此我们可以得到带修正源的标准泊松方程:

$$\Delta\chi_s = 4\pi G\rho_s. \tag{31}$$

Accordingly, the effective source ρ_s satisfies

据此，有效源 ρ_s 满足

$$\rho = f_s(\Delta)\rho_s, \tag{32}$$

where $\rho = T_{00}$ is the mass density.

其中 $\rho = T_{00}$ 为质量密度。

For a point-like source with mass $m, \rho(\vec{r}) = m\delta^{(3)}(\vec{r})$ and the effective delta source can be obtained via the Fourier transform method applied to Eq. (32). The result is

对于质量为 $m, \rho(\vec{r}) = m\delta^{(3)}(\vec{r})$ 的点源，我们可以对式 (32) 应用傅里叶变换法得到有效 δ 源，结果为

$$\rho_s(r) = \frac{m}{2\pi^2}\int_0^\infty \mathrm{d}k \frac{k\sin(kr)}{r f_s(-k^2)}, \tag{33}$$

where $r = |\vec{r}|$.

其中 $r = |\vec{r}|$ 。

The solution of Eq. (31) for the potential χ_s with the effective delta source (33) then reads

代入有效 δ 源 (33) 后，势 χ_s 满足的式 (31) 的解为

$$\chi_s(r) = G\int_\infty^r \mathrm{d}x \frac{m_s(x)}{x^2}, \tag{34}$$

where $m_s(r)$ is the effective mass function:

其中 $m_s(r)$ 是有效质量函数:

$$m_s(r) = 4\pi\int_0^r \mathrm{d}x x^2 \rho_s(x). \tag{35}$$

The formulation of the Newtonian limit of higher-derivative gravity models in terms of the effective delta source (33) has been fruitfully applied to local and non-local models (see, e.g., [27,41,42,74]). The main advantage of the method is the possibility of deriving general results about the regularity of the potentials that depend only on the behavior of the form factor in the UV regime, as we discuss in section "Resolution of Point-Like Singularities".

这种用有效 δ 源 (33) 表述高阶引力模型牛顿极限的方法，已经成功应用于局域和非局域模型 (例如参见文献 [27,41,42,74])。该方法的主要优势在于，我们可以仅依赖形状因子在紫外区域的行为，推导出关于势正则性的通用结论，我们将在“点类奇点的消解”一节对此展开讨论。

Heat Kernel Solution for the Newtonian Potentials

牛顿势的热核解

In the previous subsection, we showed how the solution for the potential χ_s can be obtained in the effective source formalism. Here, we present another integral solution for the potential, using the heat kernel approach based on the Laplace transform, as carried out in [35, 38]. This solution is particularly useful in the context of section "Mini Black Hole Formation by the Collapse of Null Shells," when dealing with the dynamic process of mini black hole formation.

在上一小节中，我们说明了如何在有效源形式体系中得到势 χ_s 的解。本文将沿用文献 [35, 38] 的工作，基于拉普拉斯变换采用热核方法，给出该势的另一种积分解。该解在处理“零壳坍缩形成迷你黑洞”章节中的迷你黑洞形成动力学过程时尤为实用。

Introducing Green's function for (27) via

我们通过以下方式为式 (27) 引入格林函数

$$f_s(\Delta)\,\Delta G_s(\vec{x},\vec{x}') = \delta^{(3)}(\vec{x}-\vec{x}'), \tag{36}$$

the integral solution for the potential is given by

势的积分解可表示为

$$\chi_s(\vec{x}) = 4\pi G \int \mathrm{d}^3x'\, G_s(\vec{x},\vec{x}')\,\rho(\vec{x}'). \tag{37}$$

Let us now assume that the inverse of the differential operator in (36) can be written as the Laplace transform of a function $h_s(s')$, that is,

现在我们假设式 (36) 中微分算符的逆可以写成函数 $h_s(s')$ 的拉普拉斯变换，即:

$$-\frac{1}{\xi f_s(-\xi)} = \int_0^\infty \mathrm{d}s'\, h_s(s')\, e^{-s'\xi}. \tag{38}$$

Then, the x -representation of Green’s function reduces to

此时，格林函数的 x 表示可简化为

$$G_s\left(\vec{x},\vec{x}'\right)=\int_0^\infty \mathrm{d}s'\, h_s\left(s'\right)\left\langle \vec{x}\left|e^{s'\Delta}\right|\vec{x}'\right\rangle, \tag{39}$$

where

其中

$$\left\langle \vec{x}\left|e^{s\Delta}\right|\vec{x}'\right\rangle = K\left(\left|\vec{x}-\vec{x}'\right|;s\right)=\frac{e^{-\frac{\left|\vec{x}-\vec{x}'\right|^2}{4s}}}{\left(4\pi s\right)^{\frac{3}{2}}} \tag{40}$$

is the heat kernel of the Laplacian.

是拉普拉斯算子的热核。

For a point-like source $\rho\left(\vec{x}\right)=m\delta^{(3)}\left(\vec{x}\right)$, the formula (37) simplifies to

对于点源 $\rho\left(\vec{x}\right)=m\delta^{(3)}\left(\vec{x}\right)$，公式 (37) 可简化为

$$\chi_s\left(r\right)=4\pi Gm\int_0^\infty \mathrm{d}s'\, h_s\left(s'\right)K\left(r;s'\right),\ s=0,2, \tag{41}$$

where $r=|\vec{x}|$. Explicit calculations using this method can be found in [15, 35, 37 , 38,40,49] and in section ”Example: Gaussian Form Factor”.

其中 $r=|\vec{x}|$。该方法的具体计算可参见文献 [15, 35, 37 , 38, 40, 49] 以及“示例: 高斯形状因子”章节。

Resolution of Point-Like Singularities

点状奇点的解决

The resolution of the Newtonian point-like singularity in non-local gravity models has been studied by several authors [2, 13 − 15, 20 − 22, 25 − 28, 32, 34, 37, 41, 47, 48 , 50, 61, 62, 74] . In most of the publications, a specific form factor was chosen and then the potentials (and, sometimes, also some curvature invariants) were evaluated, either analytically or numerically. Since there is an infinite amount of non-local form factors and of curvature invariants, it is impossible to proceed the investigation one by one. This issue was solved in [27], by using the effective source formalism (see section ”Solution for the Potentials in Terms of Effective Delta Sources”) and presenting necessary and sufficient conditions that a form factor must satisfy for the Newtonian limit to be regular - as defined in terms of gravitational potential, curvature, or curvature-derivative invariants. (We use the term curvature-derivative invariant to a generic scalar which is polynomial on curvature tensors and their covariant derivatives.)

已有多位作者 [2, 13 – 15, 20 – 22, 25 – 28, 32, 34, 37, 41, 47, 48 、50, 61, 62, 74] 研究过非局部引力模型中牛顿点状奇点的解决问题。在大多研究中，研究者会先选定一个特定形状因子，再通过解析或数值方法计算势 (有时还会计算一些曲率不变量)。由于非局部形状因子和曲率不变量都有无穷多种，不可能逐一展开研究。文献 [27] 解决了这一问题: 该文利用有效源形式论 (参见章节“用有效 δ 源表示的势的解”)，给出了牛顿极限正则化时形状因子需要满足的充要条件——正则性是通过引力势、曲率或曲率导数不变量定义的。(我们将曲率张量及其协变导数的多项式通用标量称为曲率导数不变量。)

Regarding the regularity of the potentials, if the function $f_s(-k^2)$ grows at least as fast as k^2 for large k , the smearing of the original delta source through (33) is enough to yield a potential $\chi_s(r)$ that is finite at $r = 0$ [41]. This happens even in the case of $f_s(-k^2) \sim k^2$, for which the effective delta source $\rho_s(r)$ is unbounded (see [27,28]). In most cases, however, the effective delta source is also bounded; in fact, $\rho_s(r)$ is finite if $f_s(-k^2) \sim k^4$ or faster [41]. If both χ_0 and χ_2 are regular, so are the potentials φ and ψ [see Eq. (29)]. In this spirit, one might say that the improved UV behavior of the propagator (14) regularizes the Newtonian singularity of the potentials. We stress that the resolution of this divergence is not owing to the non-locality of the theory; on the contrary, it was first noticed in the local fourth-derivative gravity [71] and later in other more general local higher-derivative models [39, 40, 64] .

关于势的正则性，若函数 $f_s(-k^2)$ 在大 k 区域增长速度不慢于 k^2 ，那么原 δ 源通过 (33) 的弥散就足以得到一个在 $r = 0$ 处有限的势 $\chi_s(r)$ [41]。即使对于 $f_s(-k^2) \sim k^2$ ，也就是有效 δ 源 $\rho_s(r)$ 无界的情况，这一结论也成立 (参见 [27,28])。但在大多数情况下，有效 δ 源本身也是有界的: 实际上当 $f_s(-k^2) \sim k^4$ 或增长更快时，$\rho_s(r)$ 就是有限的 [41]。若 χ_0 和 χ_2 都正则，那么势 φ 和 ψ 也正则 [参见式 (29)]。由此可以说，传播子 (14) 改进后的紫外行为正则化了势的牛顿奇点。我们要强调，这种发散的解决并非源于理论的非局部性；恰恰相反，它最早是在局部四阶导数引力中被发现的 [71]，后来又在其他更一般的局部高阶导数模型 [39, 40, 64] 中被讨论。

It is possible to achieve an increasing degree of regularity by including more derivatives and non-localities in the action. This can be better formulated in terms of the following definition.

我们可以通过在作用量中加入更多导数和非局部项来获得更高程度的正则性, 这一点可以通过下述定义更清晰地表述。

Definition 1 (Order of regularity of a potential [27]). If there exists a $p \in \mathbb{N}$ such that $\chi_s(r)$ is at least $2p$ -times continuously differentiable, $\chi_s^{(2p)}(r)$ is bounded and $\chi_s^{(2n+1)}(0) = 0$ for all $n \in \{0, \dots, p-1\}$, then p is called the order of regularity of $\chi_s(r)$ (equivalently, it is said that $\chi_s(r)$ is p -regular).

定义 1(势的正则性阶 [27]): 若存在 $p \in \mathbb{N}$ 使得 $\chi_s(r)$ 至少为 $2p$ 阶连续可微,且对所有 $n \in \{0, \dots, p-1\}$ 满足 $\chi_s^{(2p)}(r)$ 有界且 $\chi_s^{(2n+1)}(0) = 0$ ，则称 p 为 $\chi_s(r)$ 的正则性阶 (等价地，称 $\chi_s(r)$ 是 p 正则的)。

With this definition, as the next theorem shows, the considerations about the usual regularity of the potential can be extended to the regularity of the curvature invariants associated with the static (i.e., with $h_i \equiv 0$) metric (24).

根据这一定义，如下一定理所示，关于势通常正则性的讨论，可以推广到与静态 (即满足 $h_i \equiv 0$) 度规 (24) 相关的曲率不变量的正则性上。

Theorem 1 (Regularity of linearized curvature-derivative invariants [27]). Let the form factors be such that $f_s\left(-k^2\right)$ asymptotically grows at least as fast as k^{4+2N_s} for an integer $N_s \geq 0$ and let $N \equiv \min\{N_0, N_2\}$. Under these conditions, the potential $\chi_s(r)$ is at least (N_s+1) -regular and all the linearized curvature-derivative invariants (that is, curvature-derivative invariants calculated with the metric (24), with $h_i \equiv 0$, to the leading order in the metric perturbation) with at most $2N$ covariant derivatives are bounded.

定理 1(线性化曲率导数不变量的正则性 [27])。设形状因子满足: 对整数 $N_s \geq 0$ ，$f_s\left(-k^2\right)$ 的渐近增长速度至少不慢于 k^{4+2N_s} ， 且满足 $N \equiv \min\{N_0, N_2\}$ 。在此条件下，势 $\chi_s(r)$ 至少为 (N_s+1) 正则的，且所有含不超过 $2N$ 个协变导数的线性化曲率导数不变量 (即使用度规 (24)、在 $h_i \equiv 0$ 条件下、对度规微扰取领头阶计算得到的曲率导数不变量) 都是有界的。

Corollary 1. If $f_2\left(-k^2\right)$ and $f_0\left(-k^2\right)$ grow faster than any polynomial, then the potentials $\chi_{0,2}(r)$ are even analytic functions of r and all the curvature invariants with an arbitrary number of covariant derivatives are regular.

推论 1。若 $f_2\left(-k^2\right)$ 和 $f_0\left(-k^2\right)$ 的增长速度快于任意多项式，则势 $\chi_{0,2}(r)$ 是关于 r 的解析函数，且所有含任意个协变导数的曲率不变量都是正则的。

The proof of the theorem can be found in [27], where the regularity properties of the potential χ_s were deduced from some basic features of $f_s\left(-k^2\right)$ translated into the effective delta source (33).

该定理的证明可参见文献 [27]，其中势 χ_s 的正则性质由转化为有效 δ 源 (33) 的 $f_s\left(-k^2\right)$ 的基本性质推得。

If the potentials $\chi_{0,2}$ are only 0-regular (like in the case of the Kuz'min or Tomboulis form factors (22) that asymptotically tend to k^2), then the metric still has curvature singularities. On the other hand, if the potentials are only 1-regular, all the curvature invariants without covariant derivatives (such as R and $R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}$) are regular, but there are scalars with two covariant derivatives that diverge, such as $\Box R$ - this is the case, e.g., of the form factors (22) that tend to k^4 .

若势 $\chi_{0,2}$ 仅为 0 正则 (例如渐近趋近于 k^2 的库兹明或通布利斯形状因子 (22) 的情况)，则度规仍存在曲率奇点。另一方面，若势仅为 1 正则，所有不含协变导数的曲率不变量 (如 R 和 $R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}$) 都是正则的，但含两个协变导数的标量仍发散，例如 $\Box R$ ——趋近于 k^4 的形状因子 (22) 就是这种情况。

Hence, besides characterizing the models that have a regular Newtonian limit, the theorem also shows that the regularity of curvature invariants at $r=0$ depends on the absence of odd powers of r in the Taylor expansion of the potentials $\chi_s(r)$. The higher the first odd-order term is, the larger the set of regular scalars is. A simple example is provided by the family of scalars $\Box^n R$ (for an arbitrary n), which in the Newtonian limit reads

因此，除了刻画具有正则牛顿极限的模型，该定理还表明，$r=0$ 处曲率不变量的正则性依赖于势 $\chi_s(r)$ 的泰勒展开中不存在 r 的奇次幂。首个奇次项的阶数越高，正则标量的集合就越大。标量族 $\Box^n R$ (对任意 n) 就是一个简单例子，它在牛顿极限下可写为

$$(\Box^n R)_{\text{lin}} = 2\Delta^{n+1}\chi_0 = 2\left[\chi_0^{(2n+2)} + \frac{2(n+1)}{r}\chi_0^{(2n+1)}\right]. \tag{42}$$

For the above scalar to be regular, it suffices that the potential $\chi_0(r)$ be $(n+1)$ - regular - in other words, that $\chi_0(r)$ is $(2n+2)$ -times continuously differentiable and the limit $\lim_{r\to 0}\chi_0^{(2n+1)}(r)/r$ exists [27]. If the former condition is verified, but the latter is not, then $(\Box^n R)_{\text{lin}}$ has a singularity at $r = 0$. Only if the form factor grows faster than any polynomial [such as (21)], all the local curvature-derivative invariants are regular.

要使上述标量正则，仅需势 $\chi_0(r)$ 为 $(n+1)$ 正则——换言之，要求 $\chi_0(r)$ 是 $(2n+2)$ 次连续可微的，且极限 $\lim_{r\to 0}\chi_0^{(2n+1)}(r)/r$ 存在 [27]。若仅满足前一个条件但不满足后者，则 $(\Box^n R)_{\text{lin}}$ 在 $r = 0$ 处存在奇点。只有当形状因子增长快于任意多项式 (如 (21)) 时，所有局部曲率导数不变量才都是正则的。

Finally, we remark that Theorem 1 does not distinguish between local and non-local higher-derivative models, as it only depends on the function $f_s(-k^2)$. Therefore, the regularization of the Newtonian limit can be achieved in both types of theories, and it is not a consequence of the absence/presence of ghosts or non-localities [41]. Nevertheless, the ∞ -order regularity can only be achieved in non-local models, but not in all of them, namely, only in the ones with form factors that grow faster than any polynomial [e.g., (21)]; in this case, the effective delta source (33) and the associated Newtonian potential $\chi_s(r)$ are even analytic functions of r [27].

最后，我们指出定理 1 不区分局部与非局部高阶导数模型，因为它仅依赖函数 $f_s(-k^2)$ 。因此，牛顿极限的正则化可在两类理论中实现，这并非鬼场存在与否或非局域性带来的结果 [41]。尽管如此，∞ 阶正则性只能在非局部模型中实现，且并非所有非局部模型都满足——仅当模型中的形状因子增长快于任意多项式时才成立 [例如式 (21)]；在这种情况下，有效 δ 源 (33) 和对应的牛顿势 $\chi_s(r)$ 甚至是 r 的解析函数 [27]。

Example 1: Gaussian Form Factor

例 1: 高斯形状因子

As an explicit example, let us consider the form factor (20) with $f_0(-k^2) = f_2(-k^2) = \exp(k^2/\mu^2)$. Equation (30) is valid in this case, and the potentials can be obtained through the methods described in the previous sections; the solution reads [11, 13, 61, 70, 74]

作为一个明确示例，我们考虑满足 $f_0(-k^2) = f_2(-k^2) = \exp(k^2/\mu^2)$ 的形状因子 (20)。这种情况下方程 (30) 成立，可通过前面章节介绍的方法得到势，解为 [11, 13, 61, 70, 74]

$$\varphi(r) = -\frac{Gm}{r}\,\text{erf}\left(\frac{\mu r}{2}\right), \tag{43}$$

where $\text{erf}(x)$ is the error function. As expected from the above considerations, the gravitational potential in Eq. (43) is non-singular at $r = 0$ and tends to the finite constant value $\varphi(0) = Gm\mu/\sqrt{\pi}$, whereas in the large-distance limit $(r\mu \gg 1)$, the $1/r$ behavior of the Newtonian potential is recovered (see Fig. 1).

其中 erf(x) 是误差函数。正如上述分析预期的，式 (43) 中的引力势在 $r = 0$ 处非奇异，趋近于有限常数 $\varphi(0) = Gm\mu/\sqrt{\pi}$；而在远距离极限 $(r\mu \gg 1)$ 下，恢复为牛顿势的 $1/r$ 行为 (参见图 1)。

Moreover, since the functions $f_{0,2}\left(-k^2\right)$ grow faster than any polynomial, all the linearized curvature invariants (even those constructed with covariant derivatives of the curvatures) are bounded. For instance, we can compute the linearized Kretschmann invariant [20],

此外，由于函数 $f_{0,2}\left(-k^2\right)$ 增长比任何多项式都快，所有线性化曲率不变量 (即使是由曲率协变导数构造的不变量) 都是有界的。例如，我们可以计算线性化的克雷奇曼不变量 [20],

$$\left(R^2_{\mu\nu\alpha\beta}\right)_{\text{lin}} = \frac{G^2m^2}{3\pi r^6}e^{-\frac{\mu^2r^2}{2}}\left\{5\mu^6r^6 + 4\left[6\mu r + \mu^3r^3 - 6\sqrt{\pi}e^{\frac{\mu^2r^2}{4}}\,\text{erf}\left(\frac{\mu r}{2}\right)\right]^2\right\}, \tag{44}$$

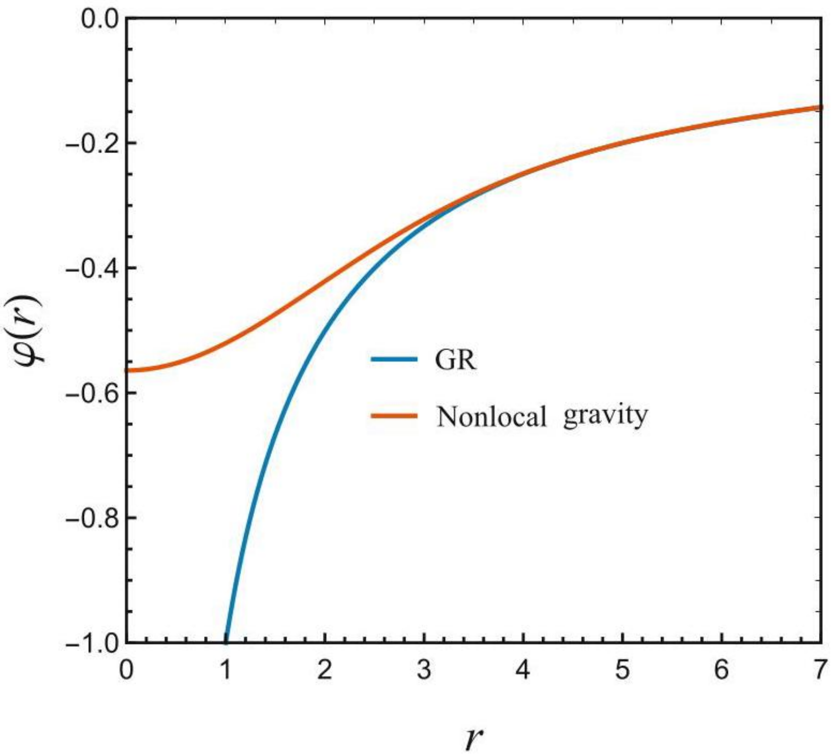

Fig. 1 Behavior of the metric potential $\varphi(r)$ for the non-local gravity model in Eq. (43) (red line) and GR (blue line). For convenience, we have set $\mu = G = m = 1$

图 1 式 (43) 中非局域引力模型 (红线) 和广义相对论 (蓝线) 的度规势 $\varphi(r)$ 的行为。为方便起见，我们设 $\mu = G = m = 1$

and explicitly verify that it is non-singular. Indeed,

并明确验证它是非奇异的。事实上，

$$\lim_{r\to 0}\left(R^2_{\mu\nu\alpha\beta}\right)_{\text{lin}} = \frac{5G^2m^2\mu^6}{3\pi}, \tag{45}$$

while in the large-distance limit, it consistently recovers the GR expression

而在远距离极限下，它一致地恢复了广义相对论的表达式

$$\left(R^2_{\mu\nu\alpha\beta}\right)_{\text{lin}} \sim \frac{48G^2m^2}{r^6}. \tag{46}$$

From a physical point of view, the regularization that we have just showed can be interpreted as if the non-local form factor smears out the point-like source at $r = 0$ on a spatial region of size $\sim 1/\mu$. In addition, in [20, 21] , it was shown that the above regular metric becomes conformally flat in the limit $r \to 0$, as the Weyl tensor vanishes at the origin; more generally, this happens provided that the potential $\chi_2(r)$ is at least 1-regular [40].

从物理角度看，我们刚才展示的正则化可以解释为: 非局域形状因子将位于 $r = 0$ 的点源弥散到大小为 $\sim 1/\mu$ 的空间区域上。此外，文献 [20, 21] 中表明，在极限 $r \to 0$ 下上述正则度规变为共形平坦，因为外尔张量在原点为零；更一般地，只要势 $\chi_2(r)$ 至少是 1-正则的，就会出现这个结果 [40]。

Similar qualitative results hold for the form factors in Eq. (21), which also grow faster than any polynomial. See, e.g., [14, 15, 34, 37] for other explicit examples of this type.

式 (21) 中的形状因子也能得到类似的定性结果，这些形状因子的增长同样比任何多项式快。这类其他具体例子可参见例如文献 [14, 15, 34, 37]。

Example 2: Kuz'min Form Factor

示例 2: 库兹明形状因子

Since the previous example dealt with a form factor that grows faster than any polynomial, here we discuss the case (presented in [27]) of a form factor that behaves like a polynomial in UV regime. To this end, let us consider the Kuz'min form factor (22) [59], for which

由于上一个示例讨论的是增长速度快于任意多项式的形状因子，本文我们将讨论 (文献 [27] 中提出的) 在紫外区表现为多项式形式的形状因子。为此，我们考虑库兹明形状因子 (22)[59]，满足

$$\gamma_s\left(-k^2\right) = \lambda_s\left[\gamma_{\text{E}} + \Gamma\left(0, k^2/\mu_s^2\right) + \ln\left(k^2/\mu_s^2\right)\right], \tag{47}$$

where μ_s is a mass parameter and $\lambda_s \in \mathbb{N}$. For large momentum, it satisfies

其中 μ_s 为质量参数，$\lambda_s \in \mathbb{N}$ 。它在大动量下满足

$$f_s\left(-k^2\right) \underset{k^2\to\infty}{\approx} e^{\lambda_s\gamma_{\text{E}}}\left(\frac{k^2}{\mu_s^2}\right)^{\lambda_s} \tag{48}$$

thus $f_s\left(-k^2\right) \sim k^{2\lambda_s}$ for k large enough. From the aforementioned results relating the UV behavior of the form factor and the order of regularity of the Newtonian potentials [27], it follows that, if $\lambda_s = 1$, the effective delta source is still singular and the associated potential is only 0-regular. On the other hand, the effective delta source is bounded for $\lambda_s \geq 2$, and the potential $\chi_s(r)$ is $(\lambda_s - 1)$ -regular.

因此当 k 足够大时可得 $f_s\left(-k^2\right) \sim k^{2\lambda_s}$ 。根据上述形状因子紫外行为与牛顿势正则性阶数的关联结果 [27]，可以推出: 若 $\lambda_s = 1$ ，有效 δ 源仍为奇异，对应的势仅为 0 阶正则。另一方面，当 $\lambda_s \geq 2$ 时，有效 δ 源有界，势 $\chi_s(r)$ 为 $(\lambda_s - 1)$ 阶正则。

These features can be viewed in Fig. 2, which displays the numerical evaluation of $\chi_s(r), \chi_s'(r), \chi_s^{(3)}(r)$, and $\chi_s^{(5)}(r)$ for the parameter $\lambda_s \in \{1,2,3,4\}$. Notice that the potential is finite for $\lambda_s = 1$, but its first derivative does not vanish at $r = 0$, indicating the singularity of the source (and of the curvature invariants). On the other hand, for $\lambda_s \geq 2$ the corresponding potential is at least $(\lambda_s - 1)$ -regular, as discussed above. More precisely, we see that $\chi_s^{(2\lambda_s-1)}(0) \neq 0$, showing that the potential is not λ_s -regular.

这些特征可参见图 2，图中给出了参数 $\lambda_s \in \{1,2,3,4\}$ 下 $\chi_s(r), \chi_s'(r), \chi_s^{(3)}(r)$ 和 $\chi_s^{(5)}(r)$ 的数值计算结果。可见当 $\lambda_s = 1$ 时势有限, 但其一阶导数在 $r = 0$ 处不消失, 说明源 (以及曲率不变量) 仍然奇异。而当 $\lambda_s \geq 2$ 时，对应的势如前文所述至少为 $(\lambda_s - 1)$ 阶正则。更准确地说，我们得到 $\chi_s^{(2\lambda_s-1)}(0) \neq 0$ ，说明势并非 λ_s 阶正则。

Similar results also hold for the form factors considered by Tomboulis [73] and Modesto [61], which behave like polynomials in the UV. See [41] for another example.

汤布鲁利斯 [73] 和莫德斯托 [61] 研究的紫外多项式型形状因子也得到了类似结果，另一个示例参见文献 [41]。

Resolution of Ringlike Singularities

环状奇点的解决

The previous discussion was focused on static solutions; here we return to the more general metric in the form (24) with nontrivial h_i and show that non-locality is also able to regularize rotating ringlike singularities. We follow the presentation of [23] and only consider the particular choice of form factor (20) such that $f_0\left(-k^2\right) = f_2\left(-k^2\right) = \exp\left(k^2/\mu^2\right)$, like in the example of section "Example 1: Gaussian Form Factor."

之前的讨论集中在静态解; 在此我们回到形式为 (24)、h_i 非平凡的更一般度规, 证明非定域性也可以正则化旋转环状奇点。我们沿用文献 [23] 的表述，仅考虑形状因子 (20) 满足 $f_0\left(-k^2\right) = f_2\left(-k^2\right) = \exp\left(k^2/\mu^2\right)$ 的特定选择，和“例 1: 高斯形状因子”小节中的例子一致。

In GR, the Kerr metric is plagued by a ring singularity that in Boyer-Lindquist coordinates is described by the equation $r^2 + a^2\cos^2\vartheta = 0$ or, in Cartesian coordinates, by $z = 0$ and $x^2 + y^2 = a^2$, where a can be thought of as the radius of the ring. We can mimic such a ring distribution by modeling the energy-momentum tensor as a Dirac delta distributed on a ring of radius a rotating with constant angular velocity ω [23]:

在广义相对论中，克尔度量存在环状奇点，在博耶-林奎斯特坐标中该奇点满足方程 $r^2+a^2\cos^2\vartheta=0$ ，在笛卡尔坐标中满足 $z=0$ 和 $x^2+y^2=a^2$ ，其中 a 可视为环的半径。我们可以通过将能量动量张量建模为分布在半径为 a 、以恒定角速度 ω 旋转的环上的狄拉克 δ 函数，来模拟这种环分布 [23]:

$$T_{00}=m\delta(z)\frac{\delta^{(2)}\left(x^2+y^2-a^2\right)}{\pi},\ T_{0i}=T_{00}v_i, \tag{49}$$

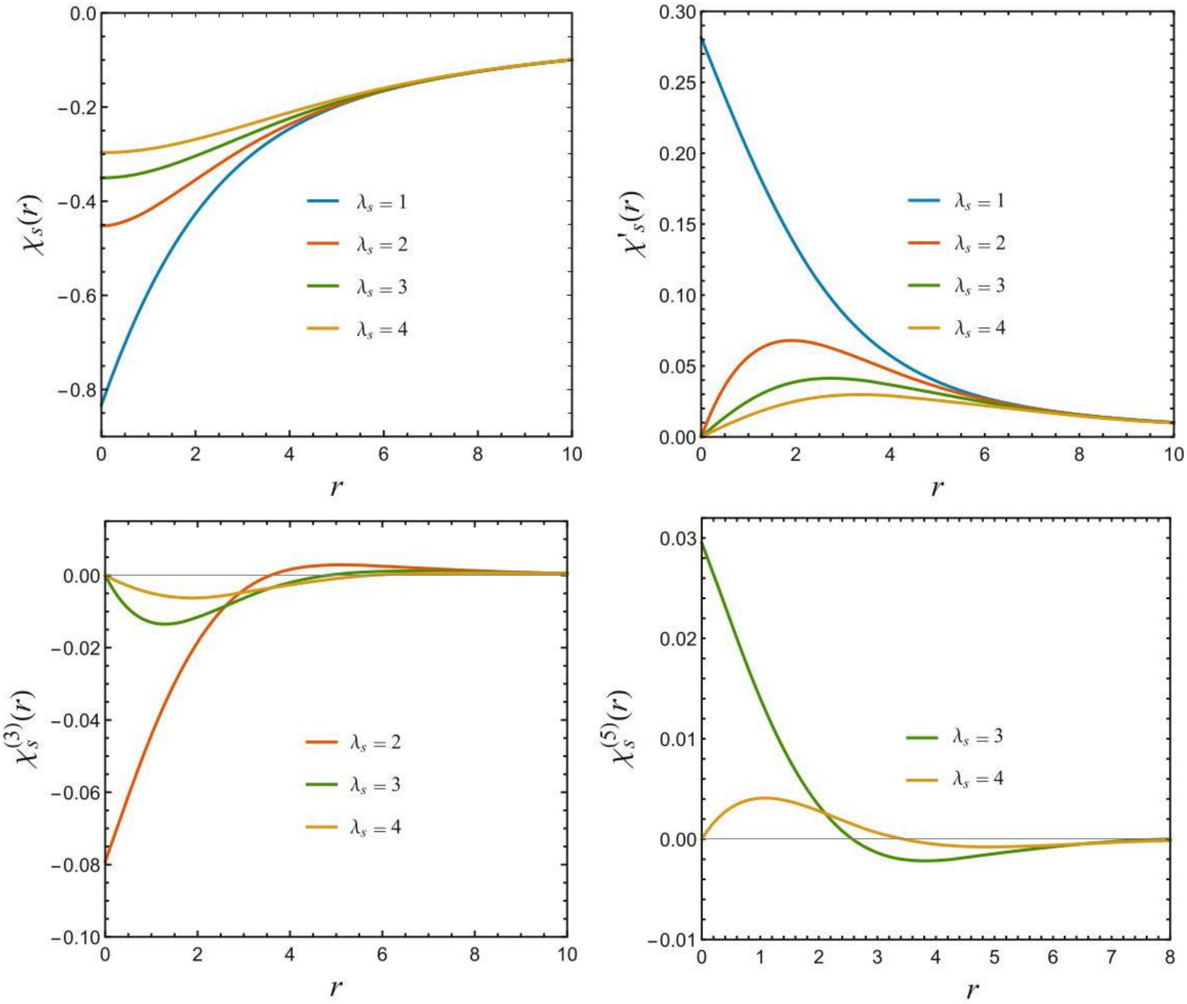

Fig. 2 Numerical evaluation of $\chi_s(r)$ and its first odd-order derivatives for the Kuz' min form factor with $\lambda_s\in\{1,2,3,4\}$ in (47); we have set $m=G=\mu_s=1$

图 2 (47) 式中库兹明形状因子 $\lambda_s\in\{1,2,3,4\}$ 对应的 $\chi_s(r)$ 及其一阶奇次导数的数值计算；我们已取 $m=G=\mu_s=1$

where v_i is the tangential velocity and its magnitude satisfies the relation $v=\omega a$. By taking the z -axis as the direction of the angular-velocity vector, we can write $v_x=-yw, v_y=xw$, and $v_z=0$. Note that, in the limit $a\to 0$, we consistently recover the expression of the point-like source because $v_i\to 0$ and $\delta^{(2)}\left(x^2+y^2-\ a^2\right)\to\delta^{(2)}\left(x^2+y^2\right)=\pi\delta(x)\delta(y)$.

其中 v_i 是切向速度，其大小满足关系 $v = \omega a$ 。取 z 轴为角速度矢量的方向，我们可以写出 $v_x = -yw, v_y = xw$ 和 $v_z = 0$ 。注意，在 $a \to 0$ 极限下，我们可以一致地得到点源的表达式，因为 $v_i \to 0$ 且 $\delta^{(2)}\left(x^2+y^2- \ a^2\right) \to \delta^{(2)}\left(x^2+y^2\right) = \pi\delta(x)\delta(y)$.

In this configuration, the off-diagonal elements of the metric (24) are nonvanishing, and the set of non-local differential equations (25) reduces to

在该构型中，度规 (24) 的非对角元非零，非定域微分方程组 (25) 可约化为

$$e^{-\Delta/\mu^2}\Delta\varphi\left(\vec{r}\right) = e^{-\Delta/\mu^2}\Delta\psi\left(\vec{r}\right) = 4Gm\delta(z)\,\delta^{(2)}\left(x^2+y^2-a^2\right),$$

$$e^{-\Delta/\mu^2}\Delta h_{0x}\left(\vec{r}\right) = -16Gm\omega y\delta(z)\,\delta^{(2)}\left(x^2+y^2-a^2\right), \tag{50}$$

$$e^{-\Delta/\mu^2}\Delta h_{0y}\left(\vec{r}\right) = 16Gm\omega x\delta(z)\,\delta^{(2)}\left(x^2+y^2-a^2\right).$$

We can use the Fourier transform method to solve the modified Poisson equations. It turns out to be useful to work in cylindrical coordinates, i.e., $x = \rho\cos\varphi$, $y = \rho\sin\varphi, z = z$. The Fourier transform of the energy-momentum tensor components read [23]

我们可以用傅里叶变换法求解修正泊松方程。在柱坐标下计算会更方便，即 $x = \rho\cos\varphi$ ，$y = \rho\sin\varphi, z = z$ 。能量动量张量分量的傅里叶变换为 [23]

$$\mathcal{T}\left[\delta(z)\,\delta^{(2)}\left(x^2+y^2-a^2\right)\right] = \pi I_0\left(ia\sqrt{k_x^2+k_y^2}\right), \tag{51}$$

$$\mathcal{T}\left[x\delta(z)\,\delta^{(2)}\left(x^2+y^2-a^2\right)\right] = \pi a\frac{k_x}{\sqrt{k_x^2+k_y^2}}I_1\left(ia\sqrt{k_x^2+k_y^2}\right), \tag{52}$$

$$\mathcal{T}\left[y\delta(z)\,\delta\left(x^2+y^2-a^2\right)\right] = \pi a\frac{k_y}{\sqrt{k_x^2+k_y^2}}I_1\left(ia\sqrt{k_x^2+k_y^2}\right), \tag{53}$$

where $\mathcal{T}[\cdots]$ stands for the Fourier transform operation and I_0 and I_1 are the modified Bessel functions of the first kind.

其中 $\mathcal{T}[\cdots]$ 代表傅里叶变换操作，I_0 和 I_1 是第一类修正贝塞尔函数。

For simplicity, let us analyze the behavior of the solution in the xy-plane (i.e. $z = 0$), where we expect the singularity to appear in GR, and work with the cylindrical radial coordinate $\rho = \sqrt{x^2+y^2}$. Therefore, by transforming back to coordinate space, the metric potentials are given by [23]

为简化分析，我们研究 xy 平面 (即 $z = 0$) 内解的行为，广义相对论中奇点预期就出现在这里，我们采用柱径向坐标 $\rho = \sqrt{x^2+y^2}$ 。因此，通过逆变换回到坐标空间，度规势由下式给出 [23]

$$\varphi(\rho) = \psi(\rho) = -Gm\int_0^\infty \mathrm{d}\zeta I_0(ia\zeta)\,I_0(i\zeta\rho)\,\mathrm{erfc}\left(\frac{\zeta}{\mu}\right), \tag{54}$$

$$h_{0x}(x,y) = 4Gm\omega a \frac{y}{\rho} H(\rho), \; h_{0y}(x,y) = -4Gm\omega a \frac{x}{\rho} H(\rho), \tag{55}$$

where $\mathrm{erfc}(z)$ is the complementary error function and

其中 $\mathrm{erfc}(z)$ 是余误差函数，且

$$H(\rho) \equiv \int_0^\infty \mathrm{d}\zeta I_1(ia\zeta) I_1(i\zeta\rho) \, \mathrm{erfc}\left(\frac{\zeta}{\mu}\right). \tag{56}$$

The above integrals cannot be solved analytically, but the integrations can be performed numerically and the solutions are plotted in Fig. 3. It shows that in GR the metric potentials are singular at $\rho = a$, as expected, whereas in the non-local gravity model under investigation, the would-be ring singularity is regularized. Also in this case, one can verify that all curvature invariants are non-singular in the entire spacetime and that the Weyl tensor vanishes at $r = 0$ [23].

上述积分无法解析求解，但可以数值计算，解绘制在图 3 中。结果显示，正如预期，广义相对论中度规势在 $\rho = a$ 处奇异，而在本文研究的非定域引力模型中，本应存在的环状奇点被正则化。同样在这个案例中，我们可以验证所有曲率不变量在整个时空中都非奇异，且外尔张量在 $r = 0$ 处为零 [23]。

Let us remark that the linearized regime holds as long as the conditions $2|\varphi| < 1$ and $|h_{0i}| < 1$ are satisfied. In particular, they are valid for any value of r if, neglecting constant factors of order one, $Gm\mu < 1$ and $Gm\mu^2\omega a^2 < 1$, respectively [23]. From the last inequality, it is evident that the angular velocity ω cannot be too large; otherwise, the linear approximation would break down. Finally, the metric outside the source was obtained using the multipole expansion also in [23], and the extension of this procedure for more general form factors was carried out in [18].

需要说明的是，只要满足条件 $2|\varphi| < 1$ 和 $|h_{0i}| < 1$ ，线性化区域就成立。若忽略一阶常数因子，分别满足 $Gm\mu < 1$ 和 $Gm\mu^2\omega a^2 < 1$ 时，该结论对任意 r 都成立 [23]。从最后一个不等式可以明显看出，角速度 ω 不能过大，否则线性近似会失效。最后，源外度规的多极展开同样在文献 [23] 中得到，文献 [18] 完成了该方法对更一般形状因子的推广。

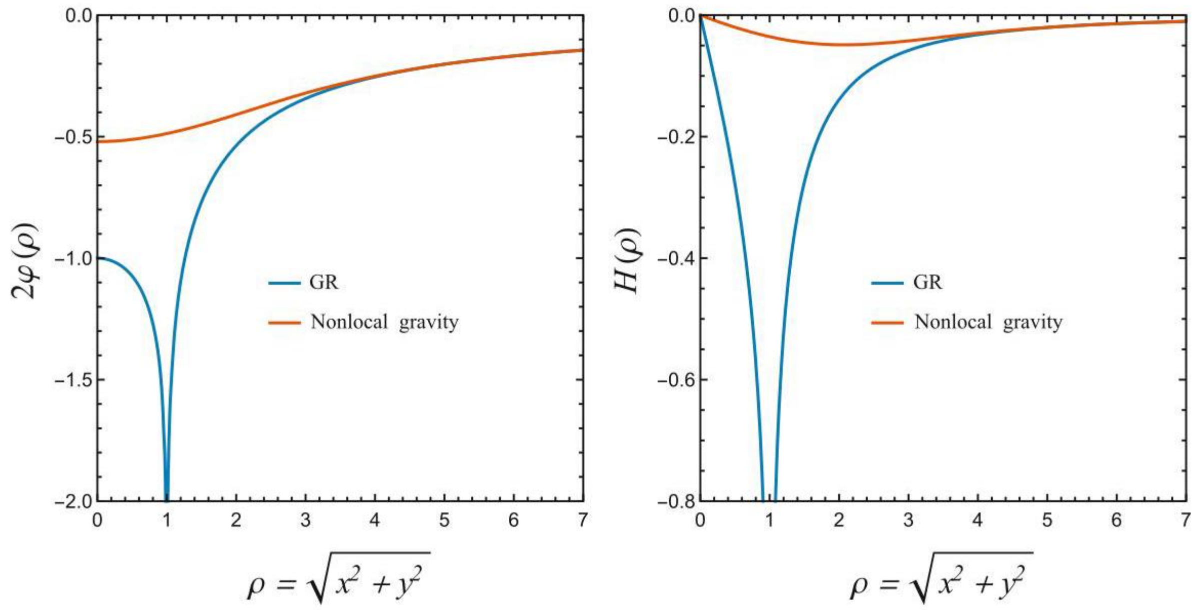

Fig. 3 Behavior of the components $-h_{00} = 2\varphi$ and $h_{0i} \sim H$ as functions of the cylindrical radius ρ , for the non-local gravity model (red line) and GR (blue line). For convenience, we have set $\mu = 1, 2Gm = 1$, and $a = 1$

图 3 分量 $-h_{00} = 2\varphi$ 和 $h_{0i} \sim H$ 随柱半径 ρ 的变化关系，红线对应非局部引力模型，蓝线对应广义相对论。为方便起见，我们设置了 $\mu = 1, 2Gm = 1$ 和 $a = 1$

Mini Black Hole Formation by the Collapse of Null Shells

由零壳坍缩形成微型黑洞

Up to this point, our considerations were restricted to linearized stationary configurations without horizons. In this section, we discuss the collapse of small-mass spherical null shells. By small mass, we mean that we continue to work with the linearized equations for the gravitational field. The interest in this scenario is the possibility of the emergence of mini black holes.

到此为止，我们的研究都局限于无视界的线性化稳态构型。在本节中，我们将讨论小质量球形零壳的坍缩过程。此处的小质量指我们仍然使用线性化方程描述引力场。研究这一情景的意义在于探究微型黑洞产生的可能性。

The gravitational field of a null shell can be built in the linearized limit using the superposition principle. By boosting the weak-field potential χ_s of a static point-like mass, the gravitational field of an ultrarelativistic particle is obtained, and taking a spherical superposition of such solutions, the linearized metric of a collapsing null shell is constructed. The formalism we follow was introduced in [38], where it was applied to non-local gravity with form factor $f_s\left(-k^2\right) = \exp\left(k^2/\mu_s^2\right)$. It was later generalized to an arbitrary form factor $f_s\left(-k^2\right)$ in [35] and applied to the case of local higher-derivative polynomial models in [35, 40] .

零壳的引力场可以在线性化极限下利用叠加原理构建。通过对静态点质量的弱场势 χ_s 进行 boost 得到极端相对论粒子的引力场，再对这类解做球形叠加，即可构造出坍缩零壳的线性化度规。我们采用的形式体系由文献 [38] 提出，该文献将其应用于带有形状因子 $f_s\left(-k^2\right)=\exp\left(k^2/\mu_s^2\right)$ 的非局域引力。之后该方法在 [35] 中被推广到任意形状因子 $f_s\left(-k^2\right)$，并在 [35, 40] 中被应用于局域高阶导数多项式模型的情况。

One of the conclusions of Refs. [35, 38, 40] is the existence of a mass gap to the formation of mini black holes if $f_s\left(-k^2\right)\sim k^2$ for large enough k . The presence of a mass gap in higher-derivative gravity models is known since the 1980s [36], and it means that a black hole can only be formed if its mass is larger than a certain critical value. This contrasts with GR, where any mass can become a black hole, provided it is concentrated in a sufficiently small region. On top of that, the curvature invariants remain bounded at $r=0$ for all theories which $f_s\left(-k^2\right)\sim k^4$ (or faster) asymptotically, assuming that the collapsing shell has a finite thickness.

文献 [35, 38, 40] 的结论之一是: 当 $f_s\left(-k^2\right)\sim k^2$ 对应足够大的 k 时，微型黑洞的形成存在质量间隙。高阶导数引力模型中存在质量间隙这一结论自 20 世纪 80 年代起就为人所知 [36]，其含义是只有当质量大于某一临界值时才可能形成黑洞。这与广义相对论形成对比——在广义相对论中，只要质量被集中在足够小的区域内，任意质量都可以形成黑洞。此外，假设坍缩壳厚度有限，对于所有渐近行为满足 $f_s\left(-k^2\right)\sim k^4$ (或更快衰减) 的理论，曲率不变量在 $r=0$ 处始终有界。

Ultrarelativistic Limit

极端相对论极限

As a first step toward the gravitational field of a collapsing shell, we consider the field associated with an ultrarelativistic point-like particle, which can be obtained by the following procedure [35,38]. First, we perform a Lorentz transformation of the metric (24) (with $h_i\equiv 0$),

作为研究坍缩壳引力场的第一步，我们考虑与极端相对论点粒子相关的场，该场可通过以下步骤得到 [35,38]。首先，我们对度规 (24)(对应 $h_i\equiv 0$) 进行洛伦兹变换,

$$t=\gamma\left(t'-\beta x'\right),\ x=\gamma\left(x'-\beta t'\right),\ \gamma\equiv\frac{1}{\sqrt{1-\beta^2}}, \tag{57}$$

which yields the metric of a moving object with velocity β in the x -direction. Then, we consider the Penrose limit, i.e., we take $\beta\to 1$ while keeping the relativistic mass

由此得到沿 x 方向以速度 β 运动的物体的度规。随后我们考虑彭罗斯极限，即令 $\beta\to 1$，同时保持该物体的相对论质量

$$M=\lim_{\gamma\to\infty}(\gamma m) \tag{58}$$

of the object fixed.

固定。

Therefore, after applying the boost (57), one gets

因此，应用 boost 变换 (57) 后可得

$$ds^2 = ds_0^2 + ds_1^2, \tag{59}$$

where

其中

$$ds_0^2 = -2\,du\,dv + dy^2 + dz^2 \tag{60}$$

is the flat spacetime metric and

为平直时空度规,

$$ds_1^2 = -\frac{\gamma^2(\varphi+\psi)}{2}\left[(1-\beta)^2\,dv^2 + (1+\beta)^2\,du^2\right] - (\varphi-\psi)\,du\,dv$$

$$-2\psi\left(dy^2 + dz^2\right) \tag{61}$$

is the first-order perturbation. In the above formulas, we introduced the advanced and retarded null coordinates:

为一阶微扰。我们在上述公式中引入了超前和推迟零坐标:

$$v = t' + x',\ u = t' - x'. \tag{62}$$

The form of the flat metric (60) remains unchanged in the limit $\beta \to 1$, while the perturbation goes to

在 $\beta \to 1$ 极限下，平直度规 (60) 的形式保持不变，而微扰变为

$$ds_1^2 = \Phi du^2,\ \text{where}\ \Phi = -4\lim_{\gamma\to\infty}\left(\gamma^2\chi_2\right). \tag{63}$$

Hence, the dominant contribution in the ultrarelativistic limit comes from the spin-2 combination of the metric potentials, $\chi_2 = (\varphi+\psi)/2$. This happens because, in this regime, the interaction between particles and the gravitational field is similar to that of photons (see, e.g., [18,43]).

因此，极端相对论极限中的主导贡献来自度规势的自旋 2 组合 $\chi_2 = (\varphi+\psi)/2$ 。出现这一结果的原因是，在该区域中，粒子与引力场的相互作用和光子的相互作用类似 (参见例如 [18,43])。

The function Φ can be evaluated through (63) by combining the heat kernel solution (41) for χ_2 and Eq. (58). Indeed, taking into account that after the boost $r^2 = \gamma^2 u^2 + y^2 + z^2$, it follows that

函数 Φ 可通过结合 χ_2 的热核解 (41) 与式 (58)，由 (63) 计算得到。事实上，考虑 boost 变换后满足 $r^2 = \gamma^2 u^2 + y^2 + z^2$ ，由此可得

$$\Phi = -4G \lim_{\gamma\to\infty} (\gamma m) \int_0^\infty \frac{\mathrm{d}s}{s} h_2(s) e^{-(y^2+z^2)/4s} \lim_{\gamma\to\infty} \frac{\gamma e^{-\gamma^2 u^2/4s}}{\sqrt{4\pi s}}. \tag{64}$$

Recalling that

我们知道

$$\lim_{\gamma\to\infty} \frac{\gamma e^{-\gamma^2 u^2/4s}}{\sqrt{4\pi s}} = \delta(u) \tag{65}$$

Eq. (64) can be written as

式 (64) 可以写为

$$\Phi = -4GMH\left(y^2 + z^2\right)\delta(u), \tag{66}$$

where we define

其中我们定义

$$H(\zeta) \equiv \int_0^{\eta^2} \frac{\mathrm{d}s}{s} h_2(s) e^{-\zeta/4s}. \tag{67}$$

Notice that in (67) it was introduced an infrared cutoff η for large s , because the integral in (64) typically has an infrared divergence, owing to the massless nature of the graviton. As discussed in [38], any change in the cutoff parameter can be absorbed into a redefinition of the coordinates. Observables, such as the curvature tensors, do not depend on η .

注意到在 (67) 中我们为大 s 引入了红外截断 η ，这是因为引力子是无质量的，(64) 中的积分通常存在红外发散。正如文献 [38] 所讨论的，截断参数的任何变化都可以被坐标重定义吸收。曲率张量这类可观测量不依赖于 η 。

The metric (63) generalizes, to higher-derivative gravity models, the Aichelburg-Sexl [5] solution in GR for the gravitational field of an ultrarelativistic massive particle without angular momentum (non-spinning gyraton). Furthermore, the extension of (63) to spinning objects can be found in [16]. (For more general considerations on gyratons in non-local theories, see [51].) Finally, the solution (63) has been used in Ref. [37] to study the black hole formation by the head-on collision of ultrarelativistic particles, with similar conclusions about the existence of mass gap as the ones derived from the collapse of null shells - which we mentioned at the beginning of the section and review in section ”Example: Gaussian Form Factor”.

度规 (63) 将广义相对论中无角动量极端相对论大质量粒子引力场的艾歇尔堡-塞克斯勒 [Aichelburg-Sexl，[5]] 解 (非自旋回旋子) 推广到了高阶导数引力模型。此外，(63) 对旋转物体的推广可在文献 [16] 中找到。(关于非定域理论中回旋子的更一般讨论，参见 [51]。) 最后，文献 [37] 使用解 (63) 研究了极端相对论粒子对撞产生黑洞的过程，其得到的质量间隙存在性结论，与我们在本节开头提到、并将在“例子: 高斯形状因子”一节中回顾的零壳坍缩结论一致。

Thin Null Shell Collapse

薄零壳坍缩

Following [35, 38], we first consider a shell with vanishing thickness. At the linearized level, the field associated with a thin null shell, or δ -shell, can be obtained by the continuous superposition of gyratons (63) spherically distributed passing through one given point O [38]. This point is the vertex of the null cone representing the shell, such that, for $t < 0$, the shell collapses toward the apex O and, for $t > 0$, it proceeds its expansion after the collapse. The energy-momentum tensor $T_{\mu\upsilon}$ associated with the shell is

沿用文献 [35, 38] 的工作，我们首先考虑厚度趋近于零的壳。在线性化层面，薄零壳 (即 δ 壳) 对应的场可以通过对经过给定点 O 、球对称分布的陀螺子 (63) 连续叠加得到 [38]。该点是代表壳的零锥的顶点，因此对于 $t < 0$ ，壳向顶点 O 坍缩，对于 $t > 0$ ，壳在坍缩后继续膨胀。与该壳关联的能量动量张量 $T_{\mu\upsilon}$ 为

$$T_{\mu\upsilon}\,\mathrm{d}x^{\mu}\mathrm{d}x^{\upsilon} = \frac{M}{4\pi r^2}\left[\delta(\upsilon)\,\mathrm{d}\upsilon^2 + \delta(u)\,\mathrm{d}u^2\right]. \tag{68}$$

It can be shown that, outside the shell, the averaged metric perturbation $\langle \mathrm{d}s_1^2\rangle$ resulting from this distribution of non-spinning gyratons is given by [38]

可以证明，在壳外部，由这种无自旋陀螺子分布产生的平均度规微扰 $\langle \mathrm{d}s_1^2\rangle$ 由下式给出 [38]

$$\langle \mathrm{d}s_1^2\rangle = -\frac{2GM}{r}H\left(r^2 - t^2\right)\left[\left(\mathrm{d}t - \frac{t}{r}\,\mathrm{d}r\right)^2 + \frac{r^2 - t^2}{2}\,\mathrm{d}\Omega^2\right],\ r \geq |t|\,, \tag{69}$$

while in its interior (in the region where $r < |t|$), the spacetime is flat, $\langle \mathrm{d}s_1^2\rangle = 0$. In (69), $H(z)$ is defined by (67), as given by the metric (63) associated with a single gyraton.

而在壳内部 ($r < |t|$ 所在区域)，时空是平直的，即 $\langle \mathrm{d}s_1^2\rangle = 0$ 。式 (69) 中，$H(z)$ 由式 (67) 定义，对应单个陀螺子的度规 (63)。

The linearized Kretschmann scalar calculated with the metric (69) is

用度规 (69) 计算得到的线性化克雷奇曼标量为

$$\left(R^2_{\mu\nu\alpha\beta}\right)_{\mathrm{lin}} = \frac{48G^2M^2\zeta^2}{r^6}Q(\zeta)\,, \tag{70}$$

$$Q(\zeta) = 2[H''(\zeta)]^2\zeta^2 + 2H'(\zeta)H''(\zeta)\zeta + [H'(\zeta)]^2, \tag{71}$$

where $\zeta = r^2 - t^2$. In this case,(70) is singular at $r = 0$ in any higher-derivative model [35, 38, 40] . However, this singularity is a consequence of the nonphysical approximation of an infinitesimally thin shell.

其中 $\zeta = r^2 - t^2$ 。在此情况下，任意高阶导数模型 [35, 38, 40] 中，式 (70) 在 $r = 0$ 处奇异。但该奇点是无限薄壳非物理近似的结果。

Thick Null Shell Collapse

厚类空壳坍缩

The metric associated with a thick null shell can be built, in the linear regime, by superposing a set of δ -shells collapsing to the same spatial point O , which is taken as the origin of the coordinate system. Suppose that the spherical collapsing null fluid is represented by a pulse with a finite time duration characterized by a density function $\rho(t)$. The total mass of the shell is

在线性区，厚类空壳的度规可以通过叠加一组坍缩到同一空间点 O 的 δ 壳构造而来，该点取为坐标系原点。假设球对称坍缩零流由密度函数 $\rho(t)$ 描述的有限持续时间脉冲表征，则壳的总质量为

$$M = \int \mathrm{d}t\rho(t), \tag{72}$$

and it collapses with the speed of light, shrinking to zero radius at the moment $t = 0$. The corresponding metric perturbation can be obtained by averaging the metric (69) of the thin null shells with respect to the density ρ [38]:

它以光速坍缩，在时刻 $t = 0$ 收缩至零半径。对应的度规扰动可以通过对薄类空壳的度规 (69) 关于密度 ρ 平均得到 [38]:

$$\langle\langle \mathrm{d}s_1^2 \rangle\rangle(t, r) = \int \mathrm{d}t'\rho(t')\langle \mathrm{d}s_1^2 \rangle(t - t', r). \tag{73}$$

Following [35, 38], we work with the simplest profile, assuming that the density $\rho(t)$ at $r = 0$ remains constant during the collapse and that it is null before and after it, namely,

依照文献 [35, 38]，我们采用最简单的分布假设: 坍缩过程中 $r = 0$ 处的密度 $\rho(t)$ 保持恒定，坍缩前后密度为零，即

$$\rho(t) = \begin{cases} 0, & \text{if } |t| > \tau/2, \\ M/\tau, & \text{if } -\tau/2 < t < \tau/2, \end{cases} \tag{74}$$

where τ is the thickness (or duration) of the shell. The distribution (74) defines specific spacetime domains; see the detailed discussion in [38]. Here we restrict considerations to the I -domain (near the point $t = r = 0$), characterized by the intersection of the incoming and the outgoing fluxes of null fluid. Formally, it is defined as the locus of the spacetime points where $r + |t| < \tau/2$. In [35, 38, 40] , it was proved that the non-singular source (74) suffices to regularize the Kretschmann scalar for theories with form factors such that $f_s\left(-k^2\right) \sim k^4$ or faster for large k .

其中 τ 是壳的厚度(或持续时间)。分布 (74) 定义了特定的时空区域，详细讨论见文献 [38]。本文我们仅讨论 I 区域(靠近点 $t = r = 0$)，该区域的特征是入射零流体通量与出射零流体通量相交。形式上，它被定义为满足 $r + |t| < \tau/2$ 的时空点的轨迹。文献 [35, 38, 40] 已证明，对于大 k 满足 $f_s\left(-k^2\right) \sim k^4$ 或衰减更快的形状因子理论，非奇异源 (74) 足以正则化克雷奇曼标量。

Taking into account that only the δ -layers which cross O at times $t' \in (t - r, t + r)$ contribute to the field inside the I -domain, it is not difficult to verify that the metric is stationary there and that Eq. (73) yields [38]

考虑到只有在时刻 $t' \in (t - r, t + r)$ 穿过 O 的 δ 层会对 I 区域内部的场产生贡献，不难验证该区域内度规是稳恒的，且由式 (73) 可得 [38]

$$\langle\langle \mathrm{d}s_{\mathrm{I}}^2 \rangle\rangle = -\frac{2GM}{\tau r}\left[c_0\, \mathrm{d}t^2 + c_2 \frac{\mathrm{d}r^2}{r^2} + \frac{1}{2}\left(c_0 r^2 - c_2\right)\mathrm{d}\Omega^2\right], r + |t| < \frac{\tau}{2}, \tag{75}$$

where

其中

$$c_n\left(r\right) \equiv \int_{-r}^{r} \mathrm{d}\xi \xi^n H\left(r^2 - \xi^2\right) \tag{76}$$

and the function $H\left(z\right)$ is defined in (67).

函数 $H\left(z\right)$ 由式 (67) 定义。

Example: Gaussian Form Factor

示例: 高斯形状因子

As an explicit example of the procedure outlined in this section, here we consider the case of the form factor (20), namely, $f_s\left(-k^2\right) = \exp\left(k^2/\mu_s^2\right)$, which was studied in [38]. The first step is to obtain the inverse Laplace transform $h_2\left(s\right)$, defined in (38), and then evaluate the function in (67). It is not difficult to show that, in this case, $h_2\left(s\right) = -\theta\left(s - \mu_2^{-2}\right)\left(\theta\left(x\right)\right.$ is the Heaviside step function) and

作为本节所述方法的一个显式示例，我们在此研究文献 [38] 中讨论过的情况，即形状因子取式 (20) 的形式，也就是 $f_s\left(-k^2\right) = \exp\left(k^2/\mu_s^2\right)$。第一步是求出式 (38) 定义的逆拉普拉斯变换 $h_2\left(s\right)$，再计算式 (67) 中的函数。不难证明，在本例中可得 $h_2\left(s\right) = -\theta\left(s - \mu_2^{-2}\right)\left(\theta\left(x\right)\right.$ is the Heaviside step function) ，且

$$H\left(\zeta\right) = \ln\left(\zeta/\eta^2\right) + \gamma_{\mathrm{E}} + E_1\left(\mu_2^2\zeta/4\right), \tag{77}$$

where γ_{E} is the Euler-Mascheroni constant and $E_1\left(x\right)$ is the exponential integral function.

其中 γ_{E} 为欧拉-马歇罗尼常数，$E_1(x)$ 为指数积分函数。

Using (77), we get for (70)

利用式 (77)，我们得到式 (70) 的结果:

$$\left(R_{\mu\nu\alpha\beta}^2\right)_{\mathrm{lin}} = \frac{3G^2M^2\mu_2^4\zeta^2}{r^6} + O\left(\zeta^3\right), \tag{78}$$

showing that as the collapse of the thin null shell proceeds, the Kretschmann scalar diverges for $r \to 0$. On the other hand, if the shell has some thickness, the curvature gets regularized. Indeed, Eqs. (76) and (77) imply the small-r behavior for the components of the metric (75),

结果表明，随着薄零壳坍缩进行，克雷奇曼标量在 $r \to 0$ 处发散。另一方面，若壳具有一定厚度，曲率则会被正则化。事实上，式 (76) 与式 (77) 暗示了度规分量 (75) 在小 r 下的行为:

$$c_0 = \frac{\mu_2^2 r^3\left(420 - 21\mu_2^2 r^2 + \mu_2^4 r^4\right)}{1260} + \ldots, \tag{79a}$$

$$c_2 = \frac{\mu_2^2 r^5\left(756 - 27\mu_2^2 r^2 + \mu_2^4 r^4\right)}{11340} + \ldots, \tag{79b}$$

so that we get for the Kretschmann invariant in the I-domain:

因此我们得到 I 区域内克雷奇曼不变量的结果:

$$\lim_{r\to 0}\left(R_{\mu\nu\alpha\beta}^2\right)_{\mathrm{lin}} = \frac{32G^2M^2\mu_2^4}{3\tau^2}. \tag{80}$$

This shows that the curvature remains finite at $r = 0$ for the thick shell model.

这说明，对于厚壳模型，曲率在 $r = 0$ 处始终有限。

Finally, let us discuss the mini black hole formation, which is related to the invariant

最后，我们来讨论与下述不变量相关的迷你黑洞形成问题:

$$(\nabla r)^2 = \frac{1}{4\varsigma} g^{\mu\upsilon}\nabla_\mu\varsigma\nabla_\upsilon\varsigma,\ \varsigma = g_{\theta\theta}. \tag{81}$$

A curve in the tr-plane such that $(\nabla r)^2 = 0$ is an apparent horizon. Using (75) and (79), one finds that, to the leading order in M, this invariant in the I-domain is given by

在 tr 平面中满足 $(\nabla r)^2 = 0$ 的曲线是表观视界。利用式 (75) 和式 (79) 可以发现，在 M 的领头阶近似下，该不变量在 I 区域的表达式为

$$(\nabla r)^2 = 1 - \frac{2M\mu_2^2 r^2}{\tau}. \tag{82}$$

Since in this domain $r < \tau$, we have

由于该区域内满足 $r < \tau$ ，我们有

$$(\nabla r)^2 > 1 - 2M\mu_2^2\tau \tag{83}$$

This relation means that, for a given μ_2 and a fixed duration τ of the pulse, the mini black hole does not form if the mass M is small enough [38].

该关系表明，对于给定 μ_2 和固定脉冲持续时间 τ ，若质量 M 足够小，则不会形成迷你黑洞 [38]。

Toward the Non-linear Regime

迈向非线性区域

The presence of higher-derivative terms in the gravitational action makes the analysis of the complete non-linear equations of motion a highly difficult task, and non-locality poses an extra obstacle, because of the infinite-order derivatives. In particular, differently from the linear limit [where any higher-derivative structure could be reduced to the form of the action (5)], the space of solutions of the complete field equations depends on the terms that are actually in the action.

引力作用量中存在高阶导数项，这使得对完整非线性运动方程的分析变得极为困难，而由于存在无穷阶导数，非局域性更带来了额外障碍。与线性极限 [任意高阶导数结构都可化简为作用量 (5) 的形式] 不同，完整场方程的解空间依赖于作用量中实际存在的项。

For example, if the Einstein-Hilbert action is enlarged only by the quadratic structures $RF_1(\Box)R$ and/or $R_{\mu\upsilon}F_2(\Box)R^{\mu\upsilon}$, the Ricci-flat solutions of GR are also vacuum solutions in the non-local model; however, if the term quadratic in the Riemann tensor is included, then those Ricci-flat spacetimes are not solutions of non-local gravity [60]. The stability of these type of solutions, imported from GR, was studied in [17, 29, 66] and in [30] for theories constructed with form factors that are functions of the Lichnerowicz operator; thermodynamic aspects of these black holes were also discussed in [31, 65, 77, 78] .

例如，如果爱因斯坦-希尔伯特作用量仅添加二次结构 $RF_1(\Box)R$ 和/或 $R_{\mu\upsilon}F_2(\Box)R^{\mu\upsilon}$ ，广义相对论的里奇平坦解在该非局域模型中同样是真空解；但如果包含黎曼张量的二次项，那么这些里奇平坦时空就不再是非局域引力的解 [60]。这种从广义相对论引入的解的稳定性，已在 [17, 29, 66] 以及文献 [30] 中针对由里奇诺维茨算符函数作为形状因子构造的理论开展了研究；这些黑洞的热力学性质也在 [31, 65, 77, 78] 中得到讨论。

It is worth noticing that this process of constructing solutions from the comparison with GR is not valid in the whole spacetime, namely, it might not correctly reflect the interaction between gravity and the matter source. For instance, while the static Schwarzschild solution in GR can be associated with a Dirac delta source (see, e.g., [7]), non-locality and higher derivatives induce a smearing of the delta source at linear level [14, 27, 37, 41, 54, 60, 61, 74] (see also section ”Resolution of Point-Like Singularities”), and their effect might be able to regularize the singularity.

值得注意的是，这种通过和广义相对论对比构造解的方法并不在全时空成立，也就是说，它可能无法正确反映引力与物质源之间的相互作用。例如，广义相对论中的静态史瓦西解可以对应狄拉克 δ 源 (参见例如文献 [7])，但非局域性和高阶导数会在线性水平对 δ 源进行弥散 [14, 27, 37, 41, 54, 60, 61, 74] (也可参见“点奇点的解决”一节)，它们的作用或许能够正则化奇点。

Since exact static black hole solutions in non-local gravity with matter sources are still an open problem, some insights were obtained from the linearized limit (as discussed above, in this chapter) and also from a non-linear approximation of the field equations, which we describe in what follows. To this end, let us consider the models of the type (19), i.e., with form factors $F_2(\Box) = -2F_1(\Box)$, so that $f_0\left(-k^2\right) = f_2\left(-k^2\right) = e^{\gamma(\Box)}$. Because of this specific form of the higher-derivative sector, the form factor gets factored together with the Einstein tensor in the field equations [61,79], namely,

由于带物质源的非局域引力中，精确静态黑洞解仍然是一个开放问题，人们已经从线性化极限 (如本章前述内容) 以及场方程的非线性近似中获得了一些见解，我们接下来将对后者进行介绍。为此，我们考虑 (19) 形式的模型，即带有形状因子 $F_2(\Box) = -2F_1(\Box)$，因此有 $f_0\left(-k^2\right) = f_2\left(-k^2\right) = e^{\gamma(\Box)}$ 。由于高阶导数部分具有这种特殊形式，形状因子会与爱因斯坦张量一同分解到场方程中 [61,79]，即

$$e^{\gamma(\Box)} G^{\mu}{}_{\upsilon} + O\left(R^2_{\dots}\right) = 8\pi G T^{\mu}{}_{\upsilon}. \tag{84}$$

In the works [8, 9, 42, 61, 62, 79] , the approximated version of this equation was considered, in the form

在研究 [8, 9, 42, 61, 62, 79] 中，研究者考虑了该方程的近似形式，写作

$$G^{\mu}{}_{\upsilon} = 8\pi G \widetilde{T}^{\mu}{}_{\upsilon}, \tag{85}$$

where

其中

$$\widetilde{T}^{\mu}{}_{\nu} = e^{-\gamma(\Box)} T^{\mu}{}_{\nu} \tag{86}$$

is an effective energy-momentum tensor.

是有效能量动量张量。

Equation (85) can be regarded as an approximation of (84) by discarding the terms $O\left(R^2\right)$ of higher order in curvatures. Furthermore, taking the d'Alembertian in (86) in its flat-spacetime form, for a point-like source $T_{00} = m\delta^{(3)}(\vec{r})$, we get

方程 (85) 可视为通过丢弃曲率中 $O\left(R^2\right)$ 高阶项得到的 (84) 近似。此外，对于点源 $T_{00} = m\delta^{(3)}(\vec{r})$，若将 (86) 中的达朗贝尔算符取为平直时空形式，我们可得

$$\widetilde{T}_{00} = \rho_{\text{eff}}, \quad \text{where} \quad \rho_{\text{eff}}(r) = \frac{m}{2\pi^2 r} \int_0^\infty \mathrm{d}k \frac{k \sin(kr)}{e^{\gamma(-k^2)}}, \tag{87}$$

just like the expression (33). Alternatively, the truncations in the original equations (84) and (87) might be compensated by imposing the conservation of the effective energy-momentum tensor, i.e., $\nabla_\mu \widetilde{T}^\mu{}_\nu = 0$ [61]. This leads to the introduction of the effective radial (p_r) and tangential (p_θ) pressures in the energy-momentum tensor:

和表达式 (33) 一致。或者，原方程 (84) 和 (87) 中的截断误差可以通过要求有效能量动量张量守恒来补偿，即 $\nabla_\mu \widetilde{T}^\mu{}_\nu = 0$ [61]。这就需要在能量动量张量中引入有效径向 (p_r) 和切向 (p_θ) 压强:

$$\widetilde{T}^\mu{}_\nu = \mathrm{diag}\left(-\rho_{\mathrm{eff}}, p_r, p_\theta, p_\theta\right). \tag{88}$$

In this spirit, generalizations of the Schwarzschild solution were obtained by using the metric ansatz

按照这一思路，研究者利用度量拟设得到了史瓦西解的推广形式

$$\mathrm{d}s^2 = -A(r)\,\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{A(r)} + r^2\,\mathrm{d}\Omega^2. \tag{89}$$

(See [42] for considerations on static spherically symmetric solutions in a more general form.) Then, making the redefinition

(关于更一般形式的静态球对称解的讨论参见文献 [42])。随后，通过重定义

$$A(r) = 1 - \frac{2G\widetilde{m}(r)}{r} \tag{90}$$

and substituting the above expressions into (85), one obtains the equations

并将上述表达式代入 (85)，可得方程

$$\frac{\mathrm{d}\widetilde{m}}{\mathrm{d}r} = 4\pi r^2 \rho_{\mathrm{eff}} = -4\pi r^2 p_r \tag{91}$$

$$\frac{\mathrm{d}p_r}{\mathrm{d}r} = \frac{2}{r}\left(p_\theta - p_r\right) - \frac{\left(p_r + \rho_{\mathrm{eff}}\right)}{2A}\frac{\mathrm{d}A}{\mathrm{d}r}, \tag{92}$$

which are solved by

其解为

$$p_r = -\rho_{\mathrm{eff}},\ p_\theta = -\rho_{\mathrm{eff}} - \frac{r\rho'_{\mathrm{eff}}}{2} \tag{93}$$

and

和

$$\widetilde{m}(r) = 4\pi \int_0^r \mathrm{d}x\, x^2 \rho_{\mathrm{eff}}(x). \tag{94}$$

Notice that $\widetilde{m}(r)$ is the same mass function as the one defined in (35).

请注意，$\tilde{m}(r)$ 是与式 (35) 中定义相同的质量函数。

For the Gaussian form factor (20), the solution is the same as the one obtained in [63, 67] , namely,

对于高斯形状因子 (20)，其解与在 [63, 67] 中得到的结果相同，即:

$$ds^2 = -\left[1 - \frac{4Gm}{r\sqrt{\pi}}\gamma\left(\frac{3}{2}, \frac{\mu^2 r^2}{4}\right)\right] dt^2 + \frac{dr^2}{1 - \frac{4Gm}{r\sqrt{\pi}}\gamma\left(\frac{3}{2}, \frac{\mu^2 r^2}{4}\right)} + r^2\, d\Omega^2, \tag{95}$$

where $\gamma(a, x)$ is the lower incomplete gamma function, or in terms of the error function,

其中 $\gamma(a, x)$ 是下不完全伽马函数，若用误差函数表示则为:

$$\gamma\left(\frac{3}{2}, \frac{\mu^2 r^2}{4}\right) = \frac{1}{2}\left[\sqrt{\pi}\,\mathrm{erf}\left(\frac{\mu r}{2}\right) - \mu r e^{-\mu^2 r^2/4}\right]. \tag{96}$$

Like the case discussed in section ”Example: Gaussian Form Factor,” there exists a mass gap for this solution to describe a black hole. In fact, if

与小节“示例: 高斯形状因子”中讨论的情况类似，该解描述黑洞存在一个质量间隙。事实上，如果

$$m < 1.9\frac{M_{\mathrm{P}}^2}{\mu}, \tag{97}$$

(we denote $1/G = M_{\mathrm{P}}^2$) the invariant $(\nabla r)^2 = A(r)$ is always positive and the metric does not have any horizon [67]. On the other hand, for values of m larger than this critical mass, the solution displays two horizons. These features are shown in Fig. 4, where we plot the graph of $A(r)$ for the two scenarios.

(我们记 $1/G = M_{\mathrm{P}}^2$) 不变量 $(\nabla r)^2 = A(r)$ 始终为正，度量不存在任何视界 [67]。另一方面，当 m 取值大于该临界质量时，解存在两个视界。这些特征如图 4 所示，我们在图中绘制了两种情形下 $A(r)$ 的图像。

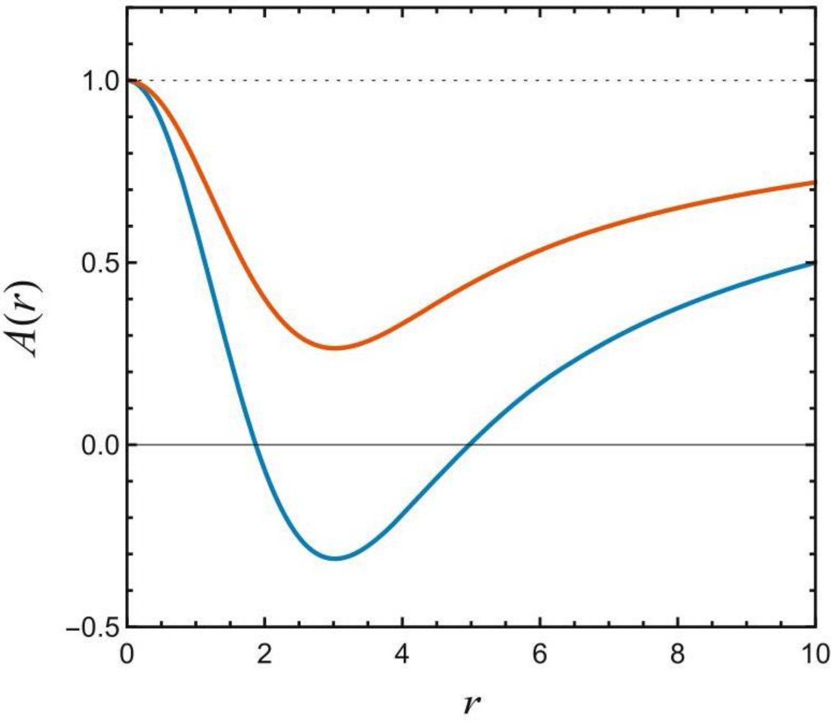

Fig. 4 Plot of $A(r)$ for $\mu = M_{\mathrm{P}} = 1$ in two situations. In blue line, with $m = 2.5$ (larger than the critical mass), the metric has two horizons at $r_- = 1.87$ and $r_+ = 4.67$. In red line, with $m = 1.4$ (smaller than the critical mass), the metric has no horizon

图 4 两种情况下，对应 $\mu = M_{\mathrm{P}} = 1$ 的 $A(r)$ 图像。蓝色线中，当 $m = 2.5$ (大于临界质量) 时，度量在 $r_- = 1.87$ 和 $r_+ = 4.67$ 处存在两个视界。红色线中，当 $m = 1.4$ (小于临界质量) 时，度量不存在视界

In what concerns the regularity of the solution, it is straightforward to verify that the metric (95) is regular, as well as the related curvature invariants [63,67]. This is expected, for the metric has a de Sitter core:

关于解的正则性，不难验证度量 (95) 及其相关曲率不变量都是正则的 [63,67]。这符合预期，因为该度量具有德西特尔核:

$$A(r) = 1 + \frac{Gm\mu^3 r^2}{3\sqrt{\pi}} + O\left(r^4\right). \tag{98}$$

Moreover, since the components of (95) are even analytic functions of r , it follows that all its local curvature-derivative invariants are finite [27,44].

此外，由于 (95) 的分量是关于 r 的偶解析函数，因此其所有局部曲率导数不变量都是有限的 [27,44]。

Similar solutions were obtained in the cases of more general exponential form factors (21) of the type $f\left(-k^2\right) = \exp\left(k^2/\mu\right)^n$, and it was shown that for larger n , the solutions can have more than two horizons [79]. The possibility of having multi-horizon black holes also occurs in the case of the Kuz'min-Tomboulis form factor (22), as numerically shown in [61,79].

对于形如 $f\left(-k^2\right)=\exp\left(k^2/\mu\right)^n$ 的更一般指数形状因子 (21)，也得到了类似解，研究表明当 n 更大时，解可以拥有两个以上的视界 [79]。正如文献 [61,79] 的数值结果所示，库兹明-通布利斯形状因子 (22) 的情况也存在多视界黑洞的可能。

Concluding Remarks

结束语

In this chapter, we reviewed linearized metric solutions in both stationary and dynamical scenarios. In the former case, we analyzed static and rotating spacetimes and showed that non-locality can regularize point-like and ringlike singularities. In the latter scenario, we showed that regularized mini black hole solutions can be found and that the formation of an apparent horizon can be avoided as long as the mass of the object is smaller than some critical value set by the scale of non-locality. Furthermore, we discussed an attempt toward finding full non-linear solutions and showed that regular solutions can be obtained by working with some simplified field equations.

本章中，我们回顾了静态和动态情景下的线性化度规解。静态情景下，我们分析了静态旋转时空，并证明非局域性可以正则化点状和环状奇点。动态情景下，我们证明可以得到正则化的迷你黑洞解，只要天体质量小于非局域性尺度设定的临界值，就可以避免表观视界形成。此外，我们讨论了寻找全非线性解的一次尝试，并证明利用若干简化场方程可以得到正则解。

Understanding the physics of non-locality at the full non-linear level remains one of the most outstanding open questions in the context of ghost-free non-local theories of gravity. In fact, it is still not entirely clear whether singularities can be really avoided in the non-linear regime or whether a horizon can form. Indeed, it has also been argued that non-locality could prevent the formation of a horizon, for any value of the mass, in such a way that black holes could be replaced by ultracompact horizonless objects in some non-local gravity models [19,52], although a rigorous proof of this statement is still lacking.

理解全非线性层面的非局域性物理，仍是无鬼非局域引力理论中最突出的开放问题之一。事实上，目前仍不完全清楚奇点能否在非线性区域真正被避免，或是视界能否形成。确实也有观点认为，在这类非局域引力模型中，非局域性可以对任意质量值都阻止视界形成，因此黑洞会被极致密无视界天体取代 [19,52]，尽管该结论仍缺乏严格证明。

Providing definite answers to these questions is challenging but at the same time very important and stimulating. Future investigations and new ideas to solve infinite-derivative non-linear differential equations are surely needed. In fact, finding a full non-linear black hole-like solution can place non-local gravity on firmer ground and lay the foundation for future phenomenological studies in astrophysics, e.g., in the context of binary mergers and gravitational waves, and thus offer a new scenario where non-local extensions of Einstein's GR can be tested and constrained.

为这些问题给出明确答案富有挑战性，同时也十分重要、令人振奋。我们无疑需要未来开展更多研究，提出解决无限导数非线性微分方程的新思路。事实上，找到全非线性类黑洞解可以让非局域引力建立在更坚实的基础上，为未来天体物理学的唯象研究 (比如双黑洞并合与引力波相关研究) 奠定基础，进而提供一个可以检验和约束爱因斯坦广义相对论非局域扩展的新场景。

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